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Direct extraction of one loop rational terms

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Abstract: We present a method for the direct extraction of rational contributions to one-loop scattering amplitudes, missed by standard four-dimensional unitarity techniques. We use generalised unitarity in $D = 4 - 2\epsilon$ dimensions to write the loop amplitudes in terms of products of massive tree amplitudes. We find that the rational terms in $4 - 2\epsilon$ dimensions can be determined from quadruple, triple and double cuts without the need for independent pentagon contributions using a massive integral basis. The additional mass-dependent integral coefficients may then be extracted from the large mass limit which can be performed analytically or numerically. We check the method by computing the rational parts of all gluon helicity amplitudes with up to six external legs. We also present a simple application to amplitudes with external massless fermions.

Keywords: Hadronic Colliders, QCD.

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1. Introduction

The main goal of the Large Hadron Collider, due to start experiments later this year, is to explore the electroweak symmetry-breaking scale and search for new physical phenomena at the TeV scale. In order to effectively achieve this goal it will be necessary to have precise predictions of backgrounds within the Standard Model and particularly for the enormous number of QCD or QCD-associated events that could mask the effects of new particles.

At the present time much progress has been made towards completing calculations for important cross sections [1]. For a selected number of processes, NNLO precision may be necessary but for the majority of events it is expected that NLO predictions will be sufficient. Because generic signals of new phenomena will be associated with the production and subsequent decay of heavy particles, the main source of backgrounds comes from multi-jet final states. One of the most important ingredients for such multiparticle NLO cross sections is the virtual matrix elements. The traditional Feynman approach to such calculations is extremely computationally intensive due to the rapid growth in the number of diagrams with the number of external legs.

On-shell techniques, pioneered by Bern, Dixon and Kosower during the mid-nineties [2, 3], offer an elegant alternative to the traditional Feynman approach. When constructing loop amplitudes from on-shell objects one works only with the physical degrees of freedom which can substantially reduce the complexity of the calculation. Loop amplitudes are reconstructed by sewing together products of on-shell tree amplitudes to extract information about the branch cuts in each channel of the momentum invariants. This information can then be used to find the coefficients of the known scalar integral basis [4 – 6]. In supersymmetric theories performing the on-shell cut in four dimensions is sufficient to reconstruct the full amplitude, whereas in QCD we will be left with additional rational terms. Knowledge of universal factorisation and use of triple as well as double cuts [7] can be used to supplement this approach and reconstruct full amplitudes in some cases to all multiplicity.

More recently on-shell techniques have seen renewed interest, via excursions into twistor space, through developments exploiting the use of complex momenta. On-shell recursive techniques at tree-level [8, 9] use basic complex analysis and universal factorisation properties to write simple relations between on-shell amplitudes. These relations have been used successfully to derive compact analytic expressions for a wide variety of processes. Since the use of complex momenta ensures that three-point amplitudes are well defined on-shell, it is also possible to refine the generalised unitarity procedure and isolate individual coefficients of the integral basis [10]. Using spinor integration techniques [11, 12] it has been possible to derive analytic expressions for the cut-constructible parts of all six-point gluon amplitudes [11, 13]. Such techniques are not restricted to massless theories and have also been developed for massive theories. They can also be applied to D-dimensional cuts [19, 20, 14 – 18], with recent applications to some examples of complete gluon amplitudes [21, 22].

Given that the integral coefficients are rational functions of the external momenta it makes sense to look for a purely algebraic procedure which avoids any explicit integration. Cutting four propagators in four dimensions completely freezes the loop integral which allows integration to be replaced by algebraic operations [10]. After integrating a generic triple cut over the on-shell delta functions a one dimensional integral still remains and it is difficult to separate any new independent functions from the residues of previously computed, higher order poles. Ossola, Papadopoulos and Pittau showed that by a special parametrisation of the loop momentum one can find a systematic way to compute these higher order terms [23]. After a subtraction of such terms at the integrand level the remaining terms can be found by solving an algebraic system of linear equations. This method has been shown to be successful in the context of Feynman diagram calculations [24 – 26] and have been implemented in a public code `CutTools` [27].

A powerful analytical approach is the one by Forde who used simple complex analysis to go beyond the OPP approach and isolate the coefficients of the scalar integrals [28]. Introducing a complex parametrisation for the loop momentum, the coefficients are then completely determined through the limiting behaviour of the products of tree amplitudes. In particular this avoids the need to solve an algebraic system of equations and leads directly

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to compact analytic expressions for the coefficients. One can understand this procedure further by observing that the non-trivial integrations are simply contour integrals in the complex plane which, after application of Cauchy's theorem, give compact descriptions of triangle coefficients [29]. This method has been generalised to accommodate arbitrary internal masses [30] and, with modifications to remove higher order poles from the complex plane, shown to be an efficient numerical tool [31]. The technique has also shown to be a powerful tool for analysing non-trivial cancellations in gravitational theories [32] and for analytic computations of multi-photon amplitudes [33, 34].

Two alternative approaches to the computation of the rational terms have been followed in recent years. The first of these is the use of loop-level recursion relations exploiting the multi-particle factorisation properties [35 – 37]. This method has been used successfully to derive analytic expressions for many helicity amplitudes up to eight final state gluons [38], as well as all-multiplicity expressions for one-loop MHV amplitudes [39, 40]. The method applies equally well to amplitudes with massive external particles where amplitudes with gluons coupling to a Higgs boson have also been derived [41 – 43]. A good review of these new techniques together can be found in reference [44]. These recursion relations have been combined with the coefficient extraction of Forde to produce an automated `C++` code, `BlackHat` [31]. Computation of amplitudes with up to eight external gluons have shown promising speed and accuracy at fixed precision.

The second method is to use D-dimensional cutting techniques which also completely determine the loop amplitude [45, 46]. Recently much progress has been made towards a numerical approaches to these techniques, based on the OPP approach. This has been shown to be much faster [47] than the current Feynman based techniques [48 – 52]. Giele, Kunszt and Melnikov [53] have used an approach using higher integer dimensions to provide a purely numerical procedure for the evaluation of the D-dimensional coefficients. This has been implemented in a `Fortran` code, `Rocket`, and used to compute gluon amplitudes with up to 20 external legs [54]. Very recently this method has also been applied to amplitudes with massive fermions [55]. Ossola, Papadopoulos and Pittau have also proposed a technique to calculate these rational parts [26] using a massive integral basis which motivates the construction we present here.

The purpose of this paper is to extend the current D-dimensional approaches to computations of rational terms by making use the complex analysis techniques used by Forde for the cut-constructible terms [28]. This allows us to compute analytic expressions for full one-loop amplitudes within a single framework and sheds further light on the efficient computations of full one-loop amplitudes. Just as in Forde's analysis, this avoids the need to solve an algebraic system of equations present in the OPP approach and leads directly to compact analytic expressions. The main simplification arising from this analysis is the separation of pentagon contributions, which vanish in the four-dimensional limit and so are eliminated from the computation. This leaves the rational terms in terms of tree amplitudes evaluated in the large-mass limit of box, triangle and bubble cuts. Alternatively this procedure can be understood as a contour integration for a complex mass parameter where the radius of the contour is taken to infinity. For the main part of this paper we concentrate on amplitudes with external gluons though the methods presented should also apply to more general external states by using the full D-dimensional tree amplitudes. This is demonstrated using a simple example involving massless external fermions. In section 3 we review the D-dimensional integral basis and the general form of the rational terms. In section 4 we describe how each of the components of the $4-2\epsilon$-dimensional basis can be determined from the large mass behaviour of four-dimensional, massive, generalised cuts. Section 5 we present some analytic examples of gluon amplitudes with up to six external legs and outline a simple numerical implementation. We then present a simple application to a four-point massless fermion amplitude in section 6 before giving our conclusions. Some additional notes on mass dependence of the one-loop integrands and general forms of the boundary expansions are given in an appendix.

2. Notation

Throughout this paper we will be considering colour-ordered helicity amplitudes. We use the standard spinor-helicity formalism to describe all momenta and external wavefunctions [56 – 61]. Two component Weyl spinors are written as

$$\lambda_\alpha(p) = |p\rangle, \qquad \tilde{\lambda}^{\dot{\alpha}}(p) = |p]. \tag{2.1}$$

The indices of the two-component spinors are raised and lowered using two totally antisymmetric tensors $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$ where $\epsilon_{12} = \epsilon_{\dot{1}\dot{2}} = 1$. The scattering amplitudes are then written in terms of spinor products defined by,

$$\lambda^\alpha(p)\lambda_\alpha(q) = \langle pq\rangle, \qquad \tilde{\lambda}_{\dot{\alpha}}(p)\tilde{\lambda}^{\dot{\alpha}}(q) = [pq]. \tag{2.2}$$

Throughout the paper all momenta are written in matrix form via contraction with the Pauli σ matrices:

$$p^\nu = \langle p|\sigma^\nu|p], \qquad\qquad p\cdot\sigma_{\alpha\dot{\alpha}} = |p\rangle_\alpha [p|_{\dot{\alpha}}, \tag{2.3}$$

where we use a shorthand notation,

$$p \equiv |p\rangle[p|. \tag{2.4}$$

For numerical evaluation of the spinor products we have used the standard approach as outlined in reference [62]. Modifications to accommodate complex and massive momenta have recently been implemented in a Mathematica package [63].

3. D-dimensional cuts and rational terms

In this section we review the D-dimensional integral basis as considered in previous constructions [53, 26, 14]. Here we focus on the connection between the $4-2\epsilon$ dimensional representation and that associated with an effective mass μ^2 [64, 45, 46].

We begin by writing a general 1-loop amplitude in terms of a D-dimensional n-point function,

$$A_n^{(1)} = \int \frac{d^D l}{(4\pi)^{D/2}} \frac{\mathcal{N}(\{p_i\}, l)}{(l^2 - m_1^2)((l-K_1)^2 - m_2^2)\dots((l+K_n)^2 - m_n^2)}. \tag{3.1}$$

The numerator function $\mathcal{N}$ contains all information from external polarisation states and wavefunctions and tensor structures from the loop momenta. Since we are concerned with computing the four dimensional limit it is useful to decompose the loop momenta as,

$$l^\nu = \bar{l}^\nu + l^\nu_{[-2\epsilon]}, \tag{3.2}$$

where $\bar{l}$ contains the four-dimensional components and $l_{[-2\epsilon]}$ contains the remaining $D-4 = -2\epsilon$ dimensional components. Using D-dimensional Passarino-Veltman reduction techniques on (3.1) allows us to reduce to a basis of scalar integral functions with rational, but D-dimensional, coefficients [53],

$$\begin{aligned} A_n^{(1),D} =& \sum_{K_5} \tilde{\mathcal{C}}_{5;K_5}(D) I^D_{5;K_5} + \sum_{K_4} \tilde{\mathcal{C}}_{4;K_4}(D) I^D_{4;K_4} \\ &+ \sum_{K_3} \mathcal{C}_{3;K_3}(D) I^D_{3;K_3} + \sum_{K_2} \mathcal{C}_{2;K_2}(D) I^D_{2;K_2} + \mathcal{C}_1(D) I_1^D, \end{aligned} \tag{3.3}$$

where we define the sets of external momenta, K_r, as the set of all ordered partitions of the n external particles into r distinct groups (the ordering is defined by that of the full amplitude $A_n^{(1)}$). We proceed by writing the amplitude in terms of an integral basis with D independent coefficients at the cost of expanding the basis of integral functions. Working in the four dimensional helicity (FDH) scheme[1] [66, 67] we keep all the external momenta and sources in four dimensions which means that the dependence on D can only arise through contracting the loop momentum with itself,

$$l^2 = \bar{l}^2 + l^2_{[-2\epsilon]} \equiv \bar{l}^2 - \mu^2. \tag{3.4}$$

We can now interpret all dimensional dependence of the coefficients in terms of their dependence on μ^2. We also know that in any renormalisable gauge theory the maximum rank of an n-point tensor integral appearing in the amplitude is n, hence for the box functions we can have up to a maximum power of μ^4 in the coefficient and up to μ^2 in the triangles and bubbles. The pentagon integral is only an independent function in D dimensions since we can find poles in the $D-4$ dimensional sub-space. As a result, the coefficient of this function in $D = 4-2\epsilon$, or residue around the extra dimensional poles, can have no dependence on ϵ. Therefore we arrive at a new basis,

$$\begin{aligned} A_n^{(1),D} =& \sum_{K_5} \tilde{C}_{5;K_5} I^D_{5;K_5} \\ &+ \sum_{K_4} C^{[0]}_{4;K_4} I^D_{4;K_4}[1] + \sum_{K_4} C^{[2]}_{4;K_4} I^D_{4;K_4}[\mu^2] + \sum_{K_4} C^{[4]}_{4;K_4} I^D_{4;K_4}[\mu^4] + \sum_{K_3} C_{3;K_3} I^D_{3;K_3}[1] \\ &+ \sum_{K_3} C^{[2]}_{3;K_3} I^D_{3;K_3}[\mu^2] + \sum_{K_2} C_{2;K_2} I^D_{2;K_2}[1] + \sum_{K_2} C^{[2]}_{2;K_2} I^D_{2;K_2}[\mu^2] + C_1 I_1^D. \end{aligned} \tag{3.5}$$

[1] It is straightforward to convert gluon amplitudes into the 't Hooft Veltman scheme by subtracting a factor of $\frac{c_\Gamma}{3} A_{\rm tree}$ from the result in the FDH scheme. Similar relations exist for amplitudes with external fermions [65].

The integrals over μ^2 can be performed by separating the integration into 4 and $D-4$ dimensional parts,

$$\int \frac{d^D l_1}{(2\pi)^D} = \int \frac{d^{-\epsilon}(\mu^2)}{(2\pi)^{-2\epsilon}} \int \frac{d^4 \bar{l}_1}{(2\pi)^4}. \tag{3.6}$$

It is fairly straightforward to write the four new integrals in terms of higher-dimensional scalar integrals using [46, 45],

$$I_n^D[\mu^{2r}] = \frac{1}{2^r} I_n^{D+2r}[1] \prod_{k=0}^{r-1} (D-4+k). \tag{3.7}$$

We also use the dimensional shift identity [68] to decompose the pentagon integrals[2]

$$I_5^D[1] = \frac{(D-4)}{2} I_5^{D+2}[1] \left(\sum_{i,j} S_{ij}^{-1} \right) + \frac{1}{2} \sum_{i=1}^{5} \sum_j S_{ij}^{-1} I_{4;K_5^{(i)}}, \tag{3.8}$$

$$S_{ij} = \frac{1}{2} \left(m_i^2 + m_j^2 - p_{ij}^2 \right). \tag{3.9}$$

In the above $K_5^{(i)}$ is one of the five sets of four partitions obtained cyclically merging two adjacent partitions of a given pentagon configuration K_5. After this is done the explicit D-dependence of the amplitude is restored [53]:

$$\begin{aligned} A_n^{(1),D} = & \frac{D-4}{2} \sum_{K_5} C_{5;K_5} I_{5;K_5}^{D+2} \\ & + \sum_{K_4} C_{4;K_4} I_{4;K_4}^D + \frac{D-4}{2} \sum_{K_4} C_{4;K_4}^{[2]} I_{4;K_4}^{D+2} + \frac{(D-4)(D-2)}{4} \sum_{K_4} C_{4;K_4}^{[4]} I_{4;K_4}^{D+4} \\ & + \sum_{K_3} C_{3;K_3} I_{3;K_3}^D + \frac{D-4}{2} \sum_{K_3} C_{3;K_3}^{[2]} I_{3;K_3}^{D+2} + \sum_{K_2} C_{2;K_2} I_{2;K_2}^D \\ & + \frac{D-4}{2} \sum_{K_2} C_{2;K_2}^{[2]} I_{2;K_2}^{D+2} + C_1 I_1^D, \end{aligned} \tag{3.10}$$

where

$$C_{4;K_4} = C_{4;K_4}^{[0]} + \sum_{i=1}^{5} \sum_j S_{ij}^{-1} \tilde{C}_{5;K_5^{(i)}}. \tag{3.11}$$

$$C_{5;K_5} = \tilde{C}_{5;K_5} \sum_{i,j} S_{ij}^{-1}. \tag{3.12}$$

After taking the 4-dimensional limit, $D \to 4-2\epsilon$, we find that the integral basis reduces to a combination of box, triangle and bubble integrals but at the cost of introducing additional rational terms,

$$A_n^{(1),4-2\epsilon} = \sum_{K_4} C_{4;K_4} I_{4;K_4}^{4-2\epsilon} + \sum_{K_3} C_{3;K_3} I_{3;K_3}^{4-2\epsilon} + \sum_{K_2} C_{2;K_2} I_{2;K_2}^{4-2\epsilon} + C_1 I_1^{4-2\epsilon} + R_n + \mathcal{O}(\epsilon). \tag{3.13}$$

[2] We refer the reader the [68] for full definitions of the quantities in eq. (3.8) although we point out that all integrals include a factor of $(-1)^{n+1}$ in their definition. The essential information is that the coefficients are just functions of the external momenta and the internal masses.

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The final step is to identify how the higher dimensional integrals in (3.10) contribute to the rational terms. This is surprisingly simple since the scalar box and scalar pentagon integrals are finite in $6-2\epsilon$ dimensions and don't contribute to the rational part. The remaining three terms, written in terms of integrals over μ, are,

$$\begin{aligned}
&I_4^{4-2\epsilon}[\mu^4] \overset{\epsilon\to 0}{\to} -\frac{1}{6},\\
&I_3^{4-2\epsilon}[\mu^2] \overset{\epsilon\to 0}{\to} -\frac{1}{2},\\
&I_2^{4-2\epsilon}[\mu^2] \overset{\epsilon\to 0}{\to} -\frac{1}{6}\left(s-3(m_1^2+m_2^2)\right).
\end{aligned} \tag{3.14}$$

The rational terms are thus given by [53, 26, 18],

$$R_n = -\frac{1}{6}\sum_{K_4} C^{[4]}_{4;K_4} - \frac{1}{2}\sum_{K_3} C^{[2]}_{3;K_3} - \frac{1}{6}\sum_{K_2}\left(K_2^2-3(m_1^2+m_2^2)\right) C^{[2]}_{2;K_2} \tag{3.15}$$

4. Extracting the integral coefficients using massive propagators

To extract the integral coefficients using generalised unitarity we need to solve the constraints which put the various propagators on-shell [28]. To generalise this to the D-dimensional case we also need to extract the μ dependence of the coefficients as defined in eq. (3.5). Since the extra dimensions in the loop momentum can be interpreted as an effective mass term, it is possible to construct the full amplitude from tree amplitudes where the internal legs have a uniform mass:

$$l_i^2 = \bar{l}_i^2 - \mu^2 = 0 \Rightarrow \bar{l}_i^2 = \mu^2. \tag{4.1}$$

This method has been used successfully within the standard unitarity cut technique [45, 46] and in conjunction with spinor integration [18, 21, 22]. Solving the system of on-shell constraints can then be achieved in exactly the same way as the four-dimensional massive case [30]. For the current study we also consider ourselves to be restricted to cases with D-dimensional scalars with massless external fermions and gauge bosons.

4.1 Box coefficients

In this section we will show that by extracting the coefficient of the D-dimensional box directly using Forde's formalism [28], we can ignore the pentagon coefficients entirely. We begin by choosing the four-momentum, $\bar{l}_1$, to be parametrised by,

$$\bar{l}_1 = aK_4^\flat + bK_1^\flat + c|K_4^\flat\rangle[K_1^\flat| + d|K_1^\flat\rangle[K_4^\flat|. \tag{4.2}$$

where the $K_{1,4}^\flat$ define a massless basis in terms of two of the external momenta:

$$\begin{aligned}
&K_4^\flat = \frac{\gamma_{14}\left(\gamma_{14}K_4 - S_4K_1\right)}{\gamma_{14}^2 - S_1S_4}, \qquad K_1^\flat = \frac{\gamma_{14}\left(\gamma_{14}K_1 - S_1K_4\right)}{\gamma_{14}^2 - S_1S_4},\\
&\gamma_{14} = K_1\cdot K_4 \pm \sqrt{(K_1\cdot K_4)^2 - S_1S_4},
\end{aligned} \tag{4.3}$$

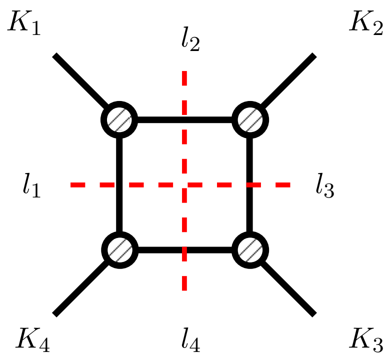


Figure 1: A general quadruple cut with loop momentum flowing clockwise and all external momenta outgoing.

for $S_i = K_i^2$.

The four on-shell constraints then fix the coefficients:

$$\begin{aligned}\bar{l}_1 &= aK_4^\flat + bK_1^\flat + c|K_4^\flat\rangle[K_1^\flat| + \frac{\gamma_{14}ab - \mu^2}{c\gamma_{14}}|K_1^\flat\rangle[K_4^\flat| \\ &= \vec{l}_1^\flat - \frac{\mu^2}{c\gamma_{14}}|K_1^\flat\rangle[K_4^\flat| \end{aligned} \tag{4.4}$$

where

$$a = \frac{S_1(S_4+\gamma_{14})}{\gamma_{14}^2 - S_1S_4}, \qquad b = -\frac{S_4(S_1+\gamma_{14})}{\gamma_{14}^2 - S_1S_4}, \qquad c_\pm = \frac{-c_1 \pm \sqrt{c_1^2 - 4c_0c_2}}{2c_2}, \tag{4.5}$$

$$c_2 = \langle K_4^\flat|K_2|K_1^\flat], \tag{4.6}$$

$$c_1 = a\langle K_4^\flat|K_2|K_4^\flat] + b\langle K_1^\flat|K_2|K_1^\flat] - S_2 - 2K_1\cdot K_2, \tag{4.7}$$

$$c_0 = \left(ab - \frac{\mu^2}{\gamma_{14}}\right)\langle K_1^\flat|K_2|K_4^\flat]. \tag{4.8}$$

For the quadruple cut we find that both solutions for γ_{14} are degenerate.

Now consider the quadruple cut:

$$\begin{aligned}&(4\pi)^{D/2}\int\frac{d^Dl_1}{(2\pi)^D}(-2\pi i)^4\prod_{i=1}^4\delta(l_i^2)A_1A_2A_3A_4 \\ &\quad= (4\pi)^{D/2}\int\frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}}\int d^4\bar{l}_1\prod_{i=1}^4\delta(\bar{l}_i^2-\mu^2)A_1A_2A_3A_4 \\ &\quad= (4\pi)^{D/2}\int\frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}}\sum_\sigma\Bigg[\mathrm{Inf}_{\mu^2}[A_1A_2A_3A_4(\bar{l}_1^\sigma)] \\ &\qquad\qquad + \sum_{\mathrm{poles}\{i\}}\frac{\mathrm{Res}_{\mu^2=\mu_i^2}(A_1A_2A_3A_4(\bar{l}_1^\sigma))}{\mu^2-\mu_i^2}\Bigg]. \end{aligned} \tag{4.9}$$

The first term encodes all the information from the boundary of the μ contour integral. The Inf operation therefore takes the form of a polynomial in μ which is cut off at some maximum power,

$$\mathrm{Inf}_{\mu^2}[f(\mu)] = \sum_{k=0}^{p} c_k \mu^{2k} \tag{4.10}$$

The second term contains information about the pentagon coefficients since it has an extra propagator in μ^2 we can identify these terms with coefficients in the basis by performing a partial fractioning in μ^2 and comparing with equation (3.10):

$$\begin{aligned}&\frac{\mathrm{Res}_{\mu^2=\mu_i^2}(A_1A_2A_3A_4(\bar{l}_1^\sigma))}{\mu^2-\mu_i^2} = \\ &\qquad \frac{\mu^2\mathrm{Res}_{\mu^2=\mu_i^2}(A_1A_2A_3A_4(\bar{l}_1^\sigma))}{\mu_i^2(\mu^2-\mu_i^2)} - \frac{\mathrm{Res}_{\mu^2=\mu_i^2}(A_1A_2A_3A_4(\bar{l}_1^\sigma))}{\mu_i^2}.\end{aligned} \tag{4.11}$$

The first term in the last equation gives the contribution to the $D+2$ dimensional pentagon integral whereas the second term combines with the μ^0 component of the boundary term to form the coefficient of the $D = 4-2\epsilon$ dimensional box integral. It is true in any case that neither of these terms contribute to the rational part and therefore we can extract the information from standard four dimensional cuts. Explicitly we find,

$$\int d^{-2\epsilon}\mu \frac{\mu^2\mathrm{Res}_{\mu^2=\mu_i^2}(A_1A_2A_3A_4)}{\mu_i^2(\mu^2-\mu_i^2)} \overset{\epsilon\to 0}{\to} 0, \tag{4.12}$$

which matches up with the vanishing of the $D+2$ dimensional pentagons from equation (3.10).

The rational contribution is therefore found by looking at the behaviour of the product of the four tree amplitudes at large values of μ,

$$\frac{i}{2}\sum_{\sigma=\pm} \mathrm{Inf}_{\mu^2}[A_1A_2A_3A_4(\bar{l}_1^\sigma)] = \sum_{k=0}^{4} \mu^k C_4^{[k]} \tag{4.13}$$

$$\Rightarrow C_4^{[4]} = \frac{i}{2}\sum_{\sigma=\pm} \mathrm{Inf}_{\mu^2}[A_1A_2A_3A_4(\bar{l}_1^\sigma)]|_{\mu^4}, \tag{4.14}$$

where we have remembered that the quadruple cut of the scalar box integral is $-i(4\pi)^{D/2}$. In the final formula, the Inf_{μ^2} operation has been restricted to the coefficient of the μ^4 term of the polynomial. We discuss the maximum possible power of μ^2 appearing in eq. (4.13) in appendix A. It is straightforward to look at the limit $\mu \to \infty$ by performing a Taylor expansion to give analytic expressions.

4.2 Triangle coefficients

We next consider the triple cut integrals[3] which, with the new dependence on μ, can be writ-

[3]We have suppressed the factors of i and π^2 in the triple and double cuts, these are implicitly put back in when writing the formulae for the coefficients.

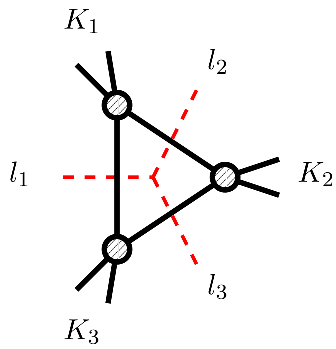


Figure 2: Momentum conventions for the triple cut; all momenta are outgoing and the loop momentum flows clockwise.

ten as,

$$
\begin{aligned}
&\int d^{-2\epsilon}\mu \int d^4\bar{l}_1 \prod_{i=1}^{3} \delta(\bar{l}_i^2 - \mu^2) A_1A_2A_3 \\
&= \int d^{-2\epsilon}\mu \int dt J_t \Bigg(\mathrm{Inf}_{\mu^2}[\mathrm{Inf}_t[A_1A_2A_3]] + \sum_i \mathrm{Inf}_{\mu^2}\left[\frac{\mathrm{Res}_{t=t_i}(A_1A_2A_3)}{t-t_i}\right] \\
&\qquad + \sum_i \frac{\mathrm{Res}_{\mu^2=\mu_i^2}(\mathrm{Inf}_t[A_1A_2A_3])}{\mu^2-\mu_i^2} + \sum_{i,j} \frac{\mathrm{Res}_{\mu^2=\mu_j^2}(\mathrm{Res}_{t=t_i}(A_1A_2A_3))}{(t-t_i)(\mu^2-\mu_j^2)} \Bigg) \qquad (4.15)
\end{aligned}
$$

Defining $K_{1,3}^\flat$ analogously to equation (4.3), we choose a loop momentum basis,

$$
l_1 = aK_3^\flat + bK_1^\flat + t|K_3^\flat\rangle[K_1^\flat| + \frac{ab\gamma_{13} - \mu^2}{\gamma_{13}t}|K_1^\flat\rangle[K_3^\flat|, \qquad (4.16)
$$

which ensures that there all integrals over t^n and $1/t^n$ vanish [28],

$$
\int dt J_t t^n = 0, \qquad \int dt J_t \frac{1}{t^n} = 0. \qquad (4.17)
$$

The second and fourth terms can be directly associated with the previously calculated scalar box coefficients. This is due the presence of an additional propagator term and the lack of t-dependence in the numerator. We can also show that the third term does not contribute to rational part by applying partial fractioning in an analogous way to the box case. Of the two terms obtained, one will be associated with the $6-2\epsilon$ dimensional boxes and the other with the $4-2\epsilon$ triangle coefficient which is cut-constructible.

Therefore the leading μ^2 dependence of the triangle functions can be completely determined through the boundary behaviour:

$$
C_3^{[2]} = \frac{1}{2}\sum_\sigma \mathrm{Inf}_{\mu^2}[\mathrm{Inf}_t[A_1A_2A_3(\bar{l}_1^\sigma)]|_{t^0}]|_{\mu^2} \qquad (4.18)
$$

One must also sum over the two solutions, σ, for the loop momentum. For the massless case when $S_1 \neq 0$ and $S_3 \neq 0$ it is sufficient to sum over the two solutions for γ_{13}. However,

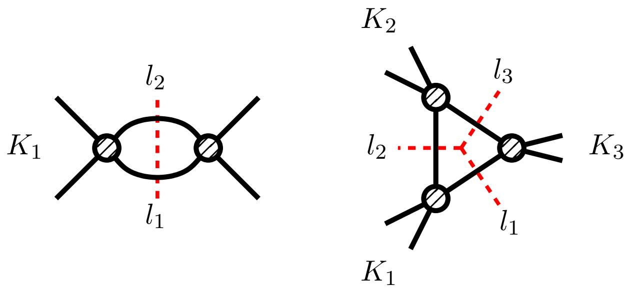


Figure 3: Pure Bubble and Triangle terms contributing to the bubble coefficients.

in general it is necessary to sum over the solution $\bar{l}_1$ given above in eq. (4.16) and conjugate solutions given by,

$$l_1^* = aK_3^\flat + bK_1^\flat + t|K_1^\flat\rangle[K_3^\flat| + \frac{ab\gamma_{13} - \mu^2}{\gamma_{13}t}|K_3^\flat\rangle[K_1^\flat|, \tag{4.19}$$

evaluated at a fixed value of γ_{13}. This applies equally well to the one and two mass triangles when one or both of S_1, S_3 vanishes and there is only a single solution for γ_{13}. As for the box case, we postpone justification of the form of eq. (4.18) to appendix A.

4.3 Bubble coefficients

Finally, we find the double cut integral using the following basis for the loop momentum:

$$\bar{l}_1 = yK_1^\flat + \frac{S_1(1-y)}{\bar{\gamma}}\chi + t|K_1^\flat\rangle[\chi| + \big(y(1-y)S_1 - \mu^2\big)\frac{|\chi\rangle[K_1^\flat|}{\bar{\gamma}t}, \tag{4.20}$$

where

$$K_1^\flat = K_1 - \frac{S_1}{\bar{\gamma}}\chi, \qquad\qquad \bar{\gamma} = 2(K_1\cdot\chi). \tag{4.21}$$

The cut integral can then be decomposed into,

$$\begin{aligned}
&\int d^{-2\epsilon}\mu \int d^4\bar{l}_1 \prod_{i=1}^{2}\delta(\bar{l}_i^2 - \mu^2)A_1A_2 \\
&= \int d^{-2\epsilon}\mu \int dtdy\ J_{t,y}\ \mathrm{Inf}_{\mu^2}[\mathrm{Inf}_t[\mathrm{Inf}_y[A_1A_2]]] \\
&\quad + \sum_i \mathrm{Inf}_{\mu^2}\left[\mathrm{Inf}_t\left[\frac{\mathrm{Res}_{y=y_i}(A_1A_2)}{y-y_i}\right]\right] + \sum_i \mathrm{Inf}_{\mu^2}\left[\frac{\mathrm{Res}_{t=t_i}(\mathrm{Inf}_y[A_1A_2])}{t-t_i}\right] \\
&\quad + \sum_{i,j}\mathrm{Inf}_{\mu^2}\left[\frac{\mathrm{Res}_{t=t_j}(\mathrm{Res}_{y=y_i}(A_1A_2))}{(y-y_i)(t-t_j)}\right] + \sum_i \frac{\mathrm{Res}_{\mu^2=\mu_i^2}(\mathrm{Inf}_t[\mathrm{Inf}_y[A_1A_2]])}{\mu^2-\mu_i^2} \\
&\quad + \sum_{i,j}\frac{\mathrm{Res}_{\mu^2=\mu_j^2}(\mathrm{Inf}_t[\mathrm{Res}_{y=y_i}(A_1A_2)])}{(\mu^2-\mu_j^2)(y-y_i)} + \sum_{i,j}\frac{\mathrm{Res}_{\mu^2=\mu_j^2}(\mathrm{Res}_{t=t_i}(\mathrm{Inf}_y[A_1A_2]))}{(\mu^2-\mu_j^2)(t-t_i)} \\
&\quad + \sum_{i,j,k}\frac{\mathrm{Res}_{\mu^2=\mu_k^2}(\mathrm{Res}_{t=t_j}(\mathrm{Res}_{y=y_i}[A_1A_2]))}{(\mu^2-\mu_k^2)(t-t_j)(y-y_i)}.
\end{aligned} \tag{4.22}$$

The last four terms contain residues in μ and as such cannot contribute to the rational terms. Of these terms, the pure $\mu^2 - \mu_i^2$ pole will also have a contribution to the $\mu = 0$ (or cut-constructible) bubble coefficient. The fourth term, in which the numerator is completely independent of y and t, can also be discarded as it has two propagator terms and can therefore only come from scalar box integrals. The remaining terms do have contributions to the bubble coefficient, and as described by Forde [28], can be determined by computing the Inf expansions for both y and t for both the double cut and triple cuts with non-vanishing integrals over t. The additional μ dependence is again determined by looking at the large μ behaviour:

$$C_2^{\text{bub}[2]} = -i\text{Inf}_{\mu^2}\text{Inf}_t[\text{Inf}_y[A_1 A_2(\bar{l}_1(y,t,\mu^2))]]|_{\mu^2,t^0,y^i\to Y_i} \tag{4.23}$$
$$C_2^{\text{tri}(K_3)[2]} = -\frac{1}{2}\sum_{\sigma=\pm}\text{Inf}_{\mu^2}\text{Inf}_t[A_1 A_2 A_3^{K_3}(\bar{l}_1(y_\sigma,t,\mu^2))]]|_{\mu^2,t^i\to T_i}. \tag{4.24}$$

with the full bubble coefficient being a sum of the pure bubble and triangle subtraction terms:

$$C_2^{[2]} = C_2^{\text{bub},[2]} + \sum_{\{K_3\}} C_2^{\text{tri}(K_3)[2]}. \tag{4.25}$$

The functions T_i and Y_i have been computed recently in references [28, 30] for arbitrary kinematics. Explicitly with a uniform internal mass we have,

$$Y_0 = 1 \quad Y_1 = \frac{1}{2} \quad Y_2 = \frac{1}{3}\left(1 - \frac{\mu^2}{S_1}\right). \tag{4.26}$$

Solving the additional on-shell constraint, $(l_1 + K_3)^2 = \mu^2$, for the triangle subtraction terms gives the two solutions for y as,

$$y_\pm = \frac{C_1 \pm \sqrt{C_1^2 + 4C_0C_2}}{2C_2}, \tag{4.27}$$

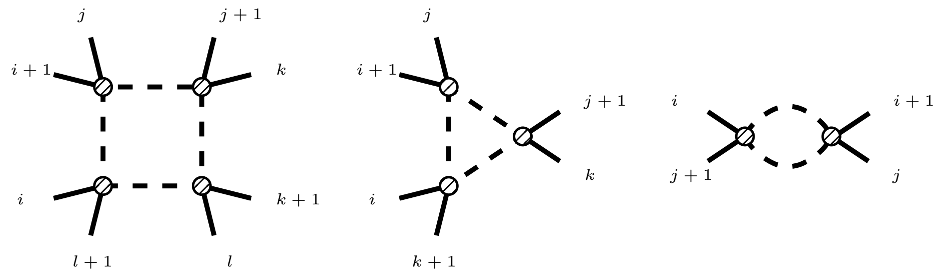
where

$$C_2 = S_1\langle\chi|K_3|K_1^\flat], \tag{4.28}$$
$$C_1 = \bar{\gamma}t\langle K_1^\flat|K_3|K_1^\flat] - S_1 t\langle\chi|K_3|\chi] + S_1\langle\chi|K_3|K_1^\flat], \tag{4.29}$$
$$C_0 = \bar{\gamma}t^2\langle K_1^\flat|K_3|\chi] - \mu^2\langle\chi|K_3|K_1^\flat] + \bar{\gamma}tS_3 + tS_1\langle\chi|K_3|\chi]. \tag{4.30}$$

The non-vanishing integrals over t are given by:

$$T_1 = -\frac{S_1\langle\chi|K_3|K_1^\flat]}{2\bar{\gamma}\Delta}, \tag{4.31}$$
$$T_2 = -\frac{3S_1\langle\chi|K_3|K_1^\flat]^2}{8\bar{\gamma}^2\Delta^2}\left(S_1S_3 + K_1\cdot K_3 S_1\right), \tag{4.32}$$
$$T_3 = -\frac{\langle\chi|K_3|K_1^\flat]^3}{48\bar{\gamma}^3\Delta^3}\bigg(15S_1S_3^2 + 30K_1\cdot K_3S_1^3S_3 + 11(K_1\cdot K_3)^2S_1^3 + 3S_1^4S_3 + 16\mu^2S_1^2\Delta\bigg), \tag{4.33}$$

where $\Delta = (K_1\cdot K_3)^2 - S_1S_3$.



$C_{4;K_{i+1,j}|K_{j+1,k}|K_{k+1,l}}$ $C_{3;K_{i+1,j}|K_{j+1,k}}$ $C_{2;K_{i+1,j}}$

Figure 4: Cut diagrams for the integral coefficients contributing to the rational parts of the n-gluon amplitudes.

5. Rational contributions to gluon amplitudes

Integral coefficients in which a single particle type circulates in the loop are particularly well suited to the method described in the previous section. Such calculations apply to give the rational terms of the all-gluon amplitudes. The supersymmetric decomposition of such amplitudes is given by,

$$A_n^g = A_n^{\mathcal{N}=4} - 4A_n^{\mathcal{N}=1} + A_n^{[s]} + N_f\left(A_n^{\mathcal{N}=1} - A_n^{[s]}\right). \tag{5.1}$$

Since supersymmetric amplitudes are cut constructible in four dimensions the rational terms in such an amplitude only appear in $A_n^{[s]}$:

$$A_n^{[s]} = C_n^{[s]} + R_n^g. \tag{5.2}$$

We can then compute the rational parts of any scalar amplitude by introducing an effective mass and evaluating the integral coefficients in four-dimensions with the tree amplitudes for massive scalars [69, 70, 19].

The rational contribution, written in terms of the massive scalar basis, is therefore,

$$\begin{aligned} R_n^g = &-\frac{1}{24}\sum_{i=1}^{n}\sum_{j=i+1}^{i-3}\sum_{k=j+1}^{i-2}\sum_{l=k+1}^{i-1} C^{[4]}_{4;K_{i+1,j}|K_{j+1,k}|K_{k+1,l}} \\ &-\frac{1}{6}\sum_{i=1}^{n}\sum_{j=i+1}^{i-2}\sum_{k=j+1}^{i-1} C^{[2]}_{3;K_{i+1,j}|K_{j+1,k}} \\ &-\frac{1}{12}\sum_{i=1}^{n}\sum_{j=i+2}^{i-2} s_{i+1,j}\, C^{[2]}_{2;K_{i+1,j}}, \end{aligned} \tag{5.3}$$

as depicted in figure 4. The arguments of the integral coefficients are considered to be taken mod(n), which accounts for the extra factors in front of the coefficients due to an over-counting in the summations. Our notation also suppresses the momentum flowing from one of the vertices of the integral since it can be inferred from momentum conservation.

5.1 Analytic expressions for the four-point amplitude

As an explicit example we present an analytic computation of the four gluon amplitude which has been considered previously with similar unitarity constructions [45, 18]. Here we only require the three- and four-point tree level amplitudes with a pair of massive scalars [69, 70]:

$$A_3(1_s, 2^-, 3_s) = i\frac{\langle 2|1|\xi_2]}{[2\xi_2]}, \qquad A_3(1_s, 2^+, 3_s) = i\frac{[2|1|\xi_2\rangle}{\langle 2\xi_2\rangle}, \tag{5.4}$$

$$A_4(1_s, 2^+, 3^+, 4_s) = i\frac{\mu^2[23]}{\langle 23\rangle\langle 2|1|2]}, \qquad A_4(1_s, 2^+, 3^-, 4_s) = i\frac{\langle 3|1|2]^2}{s_{23}\langle 2|1|2]}, \tag{5.5}$$

where μ is the mass of the scalar particles. We then construct the integrand for the quadruple cut using $K_4 = p_4, K_1 = p_1, K_2 = p_2$ which leads to a simplified solution of the general on-shell constraints from eq. (4.5):

$$a = 0, \qquad b = 0, \qquad d = -\frac{\mu^2}{cs_{41}}. \tag{5.6}$$

The fact that we only require information about the leading behaviour in μ^2 means it is sufficient to look at the leading behaviour of the coefficient c:

$$c \overset{\mu^2\to\infty}{\to} \pm\mu\sqrt{\frac{\langle K_1^\flat|K_2|K_4^\flat]}{\gamma_{14}\langle K_4^\flat|K_2|K_1^\flat]}} = \pm\mu\sqrt{\frac{\langle 1|2|4]}{\langle 4|2|1]s_{41}}}. \tag{5.7}$$

In fact, since we have a product of four tree amplitudes, we find the coefficient is only a function of c^2 and so there are no square roots appearing in the final values:

$$C_4^{[4]}(1^+, 2^+, 3^+, 4^+) = \frac{2i[21][43]}{\langle 12\rangle\langle 34\rangle}, \tag{5.8}$$

$$C_4^{[4]}(1^-, 2^+, 3^+, 4^+) = -\frac{2i\langle 14\rangle^2[42][43]}{s_{23}\langle 24\rangle\langle 34\rangle}, \tag{5.9}$$

$$C_4^{[4]}(1^-, 2^-, 3^+, 4^+) = \frac{2i\langle 12\rangle[43]}{\langle 34\rangle[21]}, \tag{5.10}$$

$$C_4^{[4]}(1^-, 2^+, 3^-, 4^+) = \frac{2i\langle 12\rangle\langle 34\rangle[42]^2}{\langle 24\rangle^2[21][43]}. \tag{5.11}$$

The triangle coefficients are particularly straightforward to evaluate since the Inf_t operation is equivalent to performing a Taylor expansion around $t = \infty$. It is helpful to choose the two massless legs of each triple cut to form the basis since this simplifies the form of the analytic expression. Let us be slightly more explicit by giving the specific details for the computation of the $C_{3;12}(1^-, 2^+, 3^+, 4^+)$ coefficient. In this channel it is easiest to choose $K_1 = p_3, K_2 = p_4$ since both S_1 and S_2 will vanish and the two solutions, one each from eq. (4.16) and (4.19), are:

$$l_1 = t|3\rangle[4| - \frac{\mu^2}{s_{12}t}|4\rangle[3|, \tag{5.12}$$

$$l_1^* = t|4\rangle[3| - \frac{\mu^2}{s_{12}t}|3\rangle[4|. \tag{5.13}$$

The product of tree amplitudes is trivial to write down:

$$A_1(-l_1, 4^+, l_2)A_2(-l_2, 1^-, 2^+, l_3)A_3(-l_3, 3^+, l_1) = 2i\frac{\langle 1|l_2|2]^2\langle 3|l_1|4]\langle 4|l_1|3]}{\langle 12\rangle\langle 34\rangle^2[12]\langle 1|l_2|2]}. \tag{5.14}$$

We are then left to insert the two solutions for l_1 and sum over the two values according to equation (5.75).

$$\text{Inf}_\mu[\text{Inf}_t[A_1A_2A_3(l_1)]]|_{\mu^2,t^0} = -2i\frac{[24]^2[34]s_{23}}{\langle 12\rangle\langle 34\rangle[12][14]^2}, \tag{5.15}$$

$$\text{Inf}_\mu[\text{Inf}_t[A_1A_2A_3(l_1^*)]]|_{\mu^2,t^0} = -2i\frac{[24]^2[34]s_{23}}{\langle 12\rangle\langle 34\rangle[12][14]^2}\left(1-\frac{s_{14}^2}{s_{24}^2}\right). \tag{5.16}$$

So, after applying momentum conservation, we find the final result:

$$C^{[2]}_{3;12}(1^-, 2^+, 3^+, 4^+) = \frac{i\left(s_{12}^2 - 2s_{24}^2\right)[32]}{\langle 23\rangle\langle 24\rangle\langle 34\rangle[21][31]}. \tag{5.17}$$

Following the same procedure for the other channels and helicity configurations quickly yields:

$$C^{[2]}_{3;23}(1^-, 2^+, 3^+, 4^+) = -\frac{i\langle 12\rangle[32]^2}{\langle 24\rangle^2[21]}, \tag{5.18}$$

$$C^{[2]}_{3;34}(1^-, 2^+, 3^+, 4^+) = -\frac{is_{12}[32][43]}{\langle 23\rangle\langle 34\rangle[31]^2}, \tag{5.19}$$

$$C^{[2]}_{3;41}(1^-, 2^+, 3^+, 4^+) = -\frac{i\left(s_{23}^2 - 2s_{24}^2\right)\langle 12\rangle}{\langle 23\rangle\langle 24\rangle^2\langle 34\rangle[41]}, \tag{5.20}$$

and for the $-+-+$ amplitude:

$$C^{[2]}_{3;12}(1^-, 2^+, 3^-, 4^+) = \frac{2is_{12}s_{24}}{\langle 24\rangle^2[31]^2}, \tag{5.21}$$

$$C^{[2]}_{3;23}(1^-, 2^+, 3^-, 4^+) = -\frac{2is_{23}\langle 13\rangle\langle 23\rangle}{\langle 24\rangle^3[41]}, \tag{5.22}$$

$$C^{[2]}_{3;34}(1^-, 2^+, 3^-, 4^+) = C^{[2]}_{3;12}(1^-, 2^+, 3^-, 4^+), \tag{5.23}$$

$$C^{[2]}_{3;41}(1^-, 2^+, 3^-, 4^+) = C^{[2]}_{3;23}(1^-, 2^+, 3^-, 4^+), \tag{5.24}$$

with all other coefficients vanishing.

For the bubble coefficients, we minimise the number of triangle subtractions by choosing $\chi = p_1$ for $K_1 = p_1 + p_2$ and $\chi = p_2$ for $K_1 = p_2 + p_3$. Again, this has the benefit of giving us results directly in terms of the external momenta and yielding simple analytic forms. The the individual solutions for y_+ and y_- in the triangle subtractions can lead to some spurious denominators and square roots. It is, therefore, beneficial to perform the sum over the two solutions algebraically. By considering the boundary behaviour at large t we can write the solution for $y_\pm$ as:

$$y_\pm = \alpha_{1,\pm}t + \alpha_{2,\pm} + \frac{1}{t}\alpha_{3,\pm} + \mathcal{O}\left(\frac{1}{t^2}\right) \tag{5.25}$$

$$\alpha_{1,\pm} = \frac{\bar{\gamma}\langle K_1^\flat|K_3|K_1^\flat] - S_1\langle\chi|K_3|\chi] \pm \sqrt{\alpha}}{2S_1\langle\chi|K_3|K_1^\flat]} \tag{5.26}$$

$$\alpha_{2,\pm} = \frac{1}{2}\left(1 \mp \frac{\sqrt{\alpha}\bar{\gamma}(S_1+S_2)}{\alpha}\right) \tag{5.27}$$

$$\alpha_{3,\pm} = \pm\frac{1}{4}\left(\frac{\sqrt{\alpha}\langle\chi|K_3|K_1^\flat](S_1-4\mu^2)}{\alpha} - \frac{\sqrt{\alpha}\bar{\gamma}^2 S_1\langle\chi|K_3|K_1^\flat](S_1+S_2)^2}{\alpha^2}\right) \tag{5.28}$$

$$\alpha = (\bar{\gamma}\langle K_1^\flat|K_3|K_1^\flat] + S_1\langle\chi|K_3|\chi])^2 - 4\bar{\gamma}^2 S_1 S_3 \tag{5.29}$$

The coefficient is then left as a function of $\alpha_{i,\pm}$ and by using sets of identities such as,

$$\frac{1}{2}\sum_{\pm}\alpha_{1,\pm}^2 = \frac{(\bar{\gamma}\langle K_1^\flat|K_3|K_1^\flat] - S_1\langle\chi|K_3|\chi])^2 + \alpha}{4S_1^2\langle\chi|K_3|K_1^\flat]^2} \tag{5.30}$$

we are able to identify terms free from square roots after the summation is performed. Such a procedure is systematic and has been performed using symbolic manipulation in FORM [71]. Therefore, after a little algebra, we quickly arrive at the expressions:

$$C^{[2]}_{2;12}(1^-,2^+,3^+,4^+) = \frac{2i\,(s_{13}-s_{23})\,\langle 13\rangle\langle 14\rangle}{\langle 12\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle^2[21]} \tag{5.31}$$

$$C^{[2]}_{2;23}(1^-,2^+,3^+,4^+) = -\frac{2is_{24}\,(s_{24}-s_{12})\,\langle 12\rangle[32]^2}{s_{23}^3\langle 24\rangle^2[21]} \tag{5.32}$$

$$C^{[2]}_{2;23}(1^-,2^-,3^+,4^+) = \frac{2i\,(2s_{12}-3s_{23})\,\langle 12\rangle^2[41]}{3\langle 14\rangle\langle 23\rangle^3[21]^2[32]} \tag{5.33}$$

$$C^{[2]}_{2;12}(1^-,2^+,3^-,4^+) = \frac{2i\,(5s_{12}+2s_{23})\,\langle 13\rangle^2}{3s_{12}^2\langle 24\rangle^2} \tag{5.34}$$

$$C^{[2]}_{2;23}(1^-,2^+,3^-,4^+) = \frac{2i\,(2s_{12}+5s_{23})\,\langle 13\rangle^2}{3s_{23}^2\langle 24\rangle^2}, \tag{5.35}$$

with all other bubble coefficients evaluating to zero.

The resulting expressions for R_4^g agree numerically with the known analytic results [45]. The form of the rational term is slightly different, and in some cases less compact, than those obtained through on-shell recursion but include both direct recursive and cut completion terms. Here we have the advantage that we do not need to calculate any completion terms for the cut-constructible parts and there are no problems associated with the factorisation in complex momenta.

5.2 The five- and six-point all-plus amplitudes

In this section we demonstrate how the technique applies to higher point amplitudes in a straightforward way. The finite helicity configuration with all gluons carrying positive helicity is a particularly simple example since there are no triangle or bubble type contributions. We only need a single five-point tree amplitude to evaluate these all-plus configurations up to the six-point level:

$$A_5^{(0)}(1_s,2^+,3^+,4^+,5_s) = \frac{i\mu^2[2|1(2+3)|4]}{\langle 23\rangle\langle 34\rangle\langle 2|1|2]\langle 4|5|4]} \tag{5.36}$$

We can again make use of eq. (5.7) to quickly determine the leading μ^2 dependence of the quadruple cut which results in:

$$R_5^g(1^+,2^+,3^+,4^+,5^+) = -\frac{1}{6}C^{[4]}_{4;12}(1^+,2^+,3^+,4^+,5^+) + \text{cyclic perms.} \tag{5.37}$$

$$C^{[4]}_{4;12}(1^+,2^+,3^+,4^+,5^+) = \frac{2i[21][43][53][54]}{\langle 12\rangle \mathrm{tr}_5(4,1,5,3)} \tag{5.38}$$

where

$$\mathrm{tr}_5(1,2,3,4) = \langle 1|234|1] - \langle 1|432|1] \tag{5.39}$$

For the six point amplitude we have three independent coefficients which also follow from a similar procedure:

$$\begin{aligned} R_6^g(1^+,2^+,3^+,4^+,5^+,6^+) = & -\frac{1}{6}C^{[4]}_{4;123}(1^+,2^+,3^+,4^+,5^+,6^+) \\ & -\frac{1}{6}C^{[4]}_{4;12|34}(1^+,2^+,3^+,4^+,5^+,6^+) \\ & -\frac{1}{12}C^{[4]}_{4;12|45}(1^+,2^+,3^+,4^+,5^+,6^+) + \text{cyclic perms.} \end{aligned} \tag{5.40}$$

$$C^{[4]}_{4;123}(1^+,2^+,3^+,4^+,5^+,6^+) = \frac{2i\left(s_{45}\langle 6|1+2|3][51][64]^2 - s_{46}\langle 5|1+2|3][54]^2[61]\right)[56]}{\langle 12\rangle\langle 23\rangle \mathrm{tr}_5(5,4,6,1)\mathrm{tr}_5(5,4,6,3)} \tag{5.41}$$

$$C^{[4]}_{4;12|34}(1^+,2^+,3^+,4^+,5^+,6^+) = \frac{2i\langle 5|1+2|6]\langle 6|1+2|5][12][43][65]^2}{\langle 12\rangle\langle 34\rangle \mathrm{tr}_5(5,2,6,1)\mathrm{tr}_5(5,4,6,3)} \tag{5.42}$$

$$C^{[4]}_{4;12|45}(1^+,2^+,3^+,4^+,5^+,6^+) = \frac{2i\left(\langle 3|1+2|3]\langle 6|1+2|6] - s_{36}s_{12}\right)[12][54][63]^2}{\langle 12\rangle\langle 45\rangle \mathrm{tr}_5(2,3,6,1)\mathrm{tr}_5(5,3,6,4)} \tag{5.43}$$

These expressions all agree with the previous analytic expressions [45, 35].

5.3 The five-point MHV amplitude

As a more involved example we present analytic expressions for the five-point MHV configuration $R_5^g(1^+,2^+,3^+,4^-,5^-)$. This amplitude has been derived previously using a string based analysis [72] and more recently using on-shell recursion relations [37]

$$\begin{aligned} R_5^g(1^+,2^+,3^+,4^-,5^-) = \sum_{k=1}^{5}\Bigg(& -\frac{1}{6}C^{[4]}_{4;k(k+1)}(1^+,2^+,3^+,4^-,5^-) \\ & -\frac{1}{2}C^{[2]}_{3;k(k+1)(k+2)}(1^+,2^+,3^+,4^-,5^-) \\ & -\frac{1}{2}C^{[2]}_{3;k(k+1)|(k+2)(k+3)}(1^+,2^+,3^+,4^-,5^-) \\ & -\frac{s_{k(k+1)}}{6}C^{[2]}_{2;k(k+1)}(1^+,2^+,3^+,4^-,5^-)\Bigg) \end{aligned} \tag{5.44}$$

$$\begin{aligned} = & -\frac{1}{6}C^{[4]}_{4;\text{total}} - \frac{1}{2}C^{[2]}_{3;\text{total}} \\ & -\sum_{k=1}^{5}\frac{s_{k(k+1)}}{6}C^{[2]}_{2;k(k+1)}(1^+,2^+,3^+,4^-,5^-) \end{aligned} \tag{5.45}$$

The computation of box contributions follows in the same way as for the all-plus configuration though the lack of symmetry means each coefficient must be computed separately. The additional tree level amplitude required is given by [69, 70],[4]

$$A_5^{(0)}(1_S,2^-,3^+,4^+,5_S)=\frac{i\langle 2|1(3+4)5|4]^2}{\langle 23\rangle\langle 34\rangle\langle 2|1|2]\langle 4|5|4][2|(3+4)5|4]}-\frac{i\mu^2[34]^3}{s_{234}[23][2|(3+4)5|4]}. \quad (5.46)$$

The sum of all five box contributions then yields the following simple result:

$$\begin{aligned}C^{[4]}_{4;\text{total}} = \sum_{k=1}^{5} C^{[4]}_{4;k(k+1)}(1^+,2^+,3^+,4^-,5^-) = \frac{2i}{\text{tr}_5(1,2,3,4)}\Bigg(&\frac{\langle 14\rangle^2\langle 25\rangle^2[53]^2[21]^3}{\langle 12\rangle^2\langle 34\rangle[43][51][52]}\\
&-\frac{2\langle 14\rangle\langle 25\rangle\langle 45\rangle[31][53][21]^2}{\langle 12\rangle\langle 34\rangle[43][51]}+\frac{\langle 45\rangle^2[31]^2[52][21]}{\langle 34\rangle[43][51]}+\frac{\langle 45\rangle[31][32][21]}{[54]}\\
&+\frac{\langle 45\rangle^2[43][53][21]}{\langle 12\rangle[54]}-\frac{2\langle 24\rangle\langle 35\rangle\langle 45\rangle[31][32]^2[41]}{\langle 15\rangle\langle 23\rangle[43][51]}+\frac{\langle 45\rangle^2[31]^2[32][42]}{\langle 15\rangle[43][51]}\\
&+\frac{\langle 24\rangle^2\langle 35\rangle^2[32]^3[41]^2}{\langle 15\rangle\langle 23\rangle^2[42][43][51]}+\frac{\langle 45\rangle^2[32][41][51]}{\langle 23\rangle[54]}\Bigg),\end{aligned} \quad (5.47)$$

while for the sum of the ten triangles we have,

$$\begin{aligned}C^{[2]}_{3;\text{total}} =& -\frac{2i\langle 24\rangle[32]^2[21]^3}{\langle 2|5+1|2]^2[42][51][52]}-\frac{i\langle 23\rangle[32]^3[21]^3}{\langle 2|5+1|2]^2[42]^2[51][52]}-\frac{i[43][21]^3}{\langle 23\rangle[42]^2[51][54]i}\\
&-\frac{i\langle 12\rangle[32]^3[21]^3}{\langle 2|5+1|2]^2[42][43][52]^2}-\frac{2i\langle 24\rangle^2\langle 25\rangle[32][21]^2}{\langle 12\rangle\langle 23\rangle\langle 2|5+1|2]^2[51]}-\frac{2i\langle 25\rangle[32]^3[21]^2}{\langle 2|5+1|2]^2[42][43][52]}\\
&-\frac{i\langle 15\rangle[53]^3[21]^2}{\langle 12\rangle^2[43][51][52]^2[54]}+\frac{2i[31][21]^2}{\langle 23\rangle[42][51][54]}-\frac{i\langle 24\rangle^2\langle 25\rangle^2[32][52][21]}{\langle 12\rangle^2\langle 23\rangle\langle 2|5+1|2]^2[51]}\\
&-\frac{2i\langle 24\rangle\langle 25\rangle^2[32]^2[21]}{\langle 12\rangle\langle 23\rangle\langle 2|5+1|2]^2[43]}-\frac{i\langle 24\rangle^2\langle 25\rangle^2[32][42][21]}{\langle 12\rangle\langle 23\rangle^2\langle 2|5+1|2]^2[43]}+\frac{i\langle 45\rangle[53]^2[21]}{\langle 12\rangle^2[51][52][54]}\\
&-\frac{i\langle 15\rangle[32][53]^2[21]}{\langle 12\rangle^2[43][52]^2[54]}+\frac{i[31]^2[21]}{\langle 23\rangle[43][51][54]}+\frac{i\langle 45\rangle[32][41]^2}{\langle 23\rangle^2[42][43][54]}+\frac{i[31]^2[32]}{\langle 12\rangle[43][51][54]}\\
&-\frac{i\langle 34\rangle[31][32][41]^2}{\langle 23\rangle^2[42][43][51][54]}+\frac{2i[31][32]^2}{\langle 12\rangle[43][52][54]}-\frac{i[32]^3[51]}{\langle 12\rangle[43][52]^2[54]}.\end{aligned} \quad (5.48)$$

The bubble coefficients are more complicated for higher point amplitudes due to the appearance of additional poles in the triangle subtraction terms. In order to give compact expressions it is convenient to leave the sum over $y_\pm$ unexpanded. We choose a value of $\chi = p_k$ for the $K_1 = p_k + p_{k+1}$ channel which ensures there are a maximum of two triangle subtraction terms. Furthermore it is useful to note that to evaluate the coefficient at the boundary of the μ contour it is sufficient to use the following values for the non-vanishing integrals:

$$Y_0 = 1 \qquad Y_1 = 0 \qquad Y_3 = -\frac{\mu^2}{3S_1} \quad (5.49)$$

$$T_1 = -\frac{S_1\langle\chi|K_3|K_1^\flat]}{2\bar\gamma\Delta} \qquad T_2 = 0 \qquad T_3 = -\frac{\mu^2 S_1\langle\chi|K_3|K_1^\flat]^3}{3\bar\gamma^3\Delta^2} \quad (5.50)$$

[4]We note that one must compensate for the different normalisation of the spinor products used in [69].

The justification for this can be seen be examining the expanded forms of the Inf operations given in appendix A.

$$
\begin{aligned}
C^{[2]}_{2;12}(1^+,2^+,3^+,4^-,5^-) &= \frac{2i([32][51]-3[31][52])[32]^2}{\langle 12\rangle^2[21][43][52]^2[54]} \\
&- 2i\sum_{\sigma=\pm}\frac{\langle 15\rangle(\langle 24\rangle-\alpha^{1;p_2}_{1,\sigma}\langle 14\rangle)^2(\alpha^{1;p_2}_{1,\sigma}\langle 15\rangle-\langle 25\rangle)[21]([31]+\alpha^{1;p_2}_{1,\sigma}[32])^2[52]}{D_{33}(\alpha^{1;p_2}_{1,\sigma},1,1+2)\langle 12\rangle^2\langle 34\rangle\langle 5|1+2|5]^2[43][51]},
\end{aligned}
\tag{5.51}
$$

$$
\begin{aligned}
C^{[2]}_{2;23}(1^+,2^+,3^+,4^-,5^-) &= \frac{2i([31][42]-3[21][43])[31]^2}{\langle 23\rangle^2[32][43]^2[51][54]} \\
&- 2i\sum_{\sigma=\pm}\frac{(\alpha^{2;p_{51}}_{1,\sigma}\langle 24\rangle-\langle 34\rangle)(\langle 35\rangle-\alpha^{2;p_{51}}_{1,\sigma}\langle 25\rangle)^2\langle 2|5+1|3]([21]+\alpha^{2;p_{51}}_{1,\sigma}[31])^2[32]}{D_{55}(\alpha^{2;p_{51}}_{1,\sigma},2,2+3)\langle 15\rangle\langle 23\rangle^2\langle 4|5+1|4]^2[42][51]},
\end{aligned}
\tag{5.52}
$$

$$
\begin{aligned}
C^{[2]}_{2;34}(1^+,2^+,3^+,4^-,5^-) &= \frac{2i\langle 24\rangle\langle 25\rangle^2\langle 35\rangle}{3\langle 12\rangle\langle 15\rangle\langle 23\rangle^4[43]} + \frac{10i\langle 25\rangle^2\langle 45\rangle}{3\langle 12\rangle\langle 15\rangle\langle 23\rangle^3[43]} - \frac{4i\langle 24\rangle[21][41]^2}{\langle 23\rangle^3[42][43][51][54]} \\
&- \frac{4i[21][31][41]}{\langle 23\rangle^2[42][43][51][54]} + \frac{2i[21][32][41]^2}{\langle 23\rangle^2[42]^2[43][51][54]} + \frac{2i[21][41]^2[53]}{\langle 23\rangle^2[42][43][51][54]^2} \\
&- 2i\sum_{\sigma=\pm}\Bigg(\frac{(\alpha^{3;p_{12}}_{1,\sigma}\langle 35\rangle-\langle 45\rangle)\langle 3|1+2|4][21][43](\alpha^{3;p_{12}}_{1,\sigma})^4}{D_{11}(\alpha^{3;p_{12}}_{1,\sigma},3,3+4)\langle 12\rangle\langle 5|1+2|5]^2[53]} \\
&\qquad + \frac{\alpha^{3;p_2}_{1,\sigma}D_{51}(\alpha^{3,p_2}_{1,\sigma},3,3+4)^2\langle 3|4|2][42](\alpha^{3;p_2}_{1,\sigma})^3}{D_{55}(\alpha^{3;p_2}_{1,\sigma},3,3+4)\langle 15\rangle\langle 2|3+4|2]^2[51]} \\
&\qquad + \frac{8D_{51}(\alpha^{3;p_2}_{1,\sigma},3,3+4)^2\langle 23\rangle^2\langle 34\rangle[42]^3([32]+\alpha^{3;p_2}_{1,\sigma}[42])(\alpha^{3;p_2}_{1,\sigma})^3}{3D_{55}(\alpha^{3;p_2}_{1,\sigma},3,3+4)s_{34}\langle 15\rangle\langle 2|3+4|2]^4[51]} \\
&\qquad + \frac{2D_{51}(\alpha^{3;p_2}_{1,\sigma},3,3{+}4)D'_{51}(\alpha^{3;p_2}_{1,\sigma},\alpha^{3,p_2}_{1,\sigma},3,3{+}4)\langle 34\rangle[42]([32]{+}\alpha^{3;p_2}_{1,\sigma}[42])(\alpha^{3;p_2}_{1,\sigma})^3}{D_{55}(\alpha^{3;p_2}_{1,\sigma},3,3+4)\langle 15\rangle\langle 2|3+4|2]^2[51]} \\
&\qquad - \frac{D_{51}(\alpha^{3;p_2}_{1,\sigma},3,3{+}4)^2D'_{55}(\alpha^{3;p_2}_{1,\sigma},\alpha^{3;p_2}_{1,\sigma},3,3{+}4)\langle 34\rangle[42]([32]{+}\alpha^{3;p_2}_{1,\sigma}[42])(\alpha^{3;p_2}_{1,\sigma})^3}{D_{55}(\alpha^{3;p_2}_{1,\sigma},3,3+4)^2\langle 15\rangle\langle 2|3+4|2]^2[51]} \\
&\qquad - \frac{\alpha^{3;p_2}_{1,\sigma}D_{51}(\alpha^{3;p_2}_{1,\sigma},3,3+4)^2s_{34}[42]([32]+\alpha^{3,p_2}_{1,\sigma}[42])(\alpha^{3;p_2}_{1,\sigma})^2}{D_{55}(\alpha^{3;p_2}_{1,\sigma},3,3+4)\langle 15\rangle\langle 2|3+4|2]^2[43][51]} \\
&\qquad + \frac{2D_{51}(\alpha^{3;p_2}_{1,\sigma},3,3+4)^2\left(2\alpha^{3;p_2}_{1,\sigma}\alpha^{3,p_2}_{1,\sigma}s_{34}+1\right)[42]([32]+\alpha^{3,p_2}_{1,\sigma}[42])\alpha^{3,p_2}_{1,\sigma}}{D_{55}(\alpha^{3;p_2}_{1,\sigma},3,3+4)\langle 15\rangle\langle 2|3+4|2]^2[43][51]}\Bigg),
\end{aligned}
\tag{5.53}
$$

$$
C^{[2]}_{2;45}(1^+,2^+,3^+,4^-,5^-) = 0, \tag{5.54}
$$

$$
C^{[2]}_{2;51}(1^+,2^+,3^+,4^-,5^-) = C^{[2]}_{2;34}(3^+,2^+,1^+,5^-,4^-), \tag{5.55}
$$

where we define the following functions,

$$
\alpha^{k;K_3}_{1,\pm} = \frac{\langle k|k+1|k]\langle k+1|K_3|k+1]-s_{k(k+1)}\langle k|K_3|k]\mp\alpha}{2s_{k(k+1)}\langle k|K_3|k+1]}, \tag{5.56}
$$

$$
\alpha^{k;K_3}_{3,\pm} = \mp\frac{\sqrt{\alpha}}{\alpha}\langle k|K_3|k+1], \tag{5.57}
$$

$$D_{xy}(\alpha,p,q) = -\alpha^2\langle xp\rangle[q^\flat y] + \alpha\left(\langle x|q^\flat|y] - \frac{q^2}{\langle p|q|p]}\langle x|p|y]\right) + \langle xq^\flat\rangle[py], \tag{5.58}$$

$$D'_{xy}(\alpha,\beta,p,q) = -\left(\frac{1}{2q\cdot p} + 2\alpha\beta\right)\langle xp\rangle[q^\flat y] + \beta\left(\langle x|q^\flat|y] - \frac{q^2}{2q\cdot p}\langle x|p|y]\right), \tag{5.59}$$

$$\text{where } q^\flat = q - \frac{q^2}{2p\cdot q}p.$$

The function α is given in equation (5.29). These expressions have been checked numerically against the known results of [72].

5.4 The six-point $-+-+-+$ amplitude

As a final analytic example we consider the NMHV six-point amplitude with alternating helicities, $R_6^g(1^-,2^+,3^-,4^+,5^-,6^+)$. This amplitude has been considered in previous analytic studies based on Feynman diagrams [73] as well as more recent numerical unitarity approaches [31, 53]. Here the analytic expression can be reduced to a set of seven independent coefficients through the symmetries of the external helicities. We first define three transformation identities:

$$\alpha_l : (1,2,3,4,5,6) \mapsto (l+1,l+2,l+3,l+4,l+5,l+6) \tag{5.60}$$

$$\alpha_l^\dagger : (1,2,3,4,5,6) \mapsto (l+1,l+2,l+3,l+4,l+5,l+6)|_{\langle\rangle\leftrightarrow[]} \tag{5.61}$$

$$\beta : (1,2,3,4,5,6) \mapsto (6,5,4,3,2,1) \tag{5.62}$$

The rational contribution to this amplitude can then be written as,

$$\begin{aligned}
R_6^g(1^-,2^+,3^-,4^+,5^-,6^+) = -\frac{1}{6}\sum_{k=0,2,4}\Bigg(&\alpha_k\left(C^{[4]}_{4;123}\right) + \alpha_k\left(C^{[4]}_{4;12|34}\right)\\
&+\alpha_k\left(C^{[4]}_{4;12|45}\right) + \alpha^\dagger_{k+1}\left(C^{[4]}_{4;123}\right) + \alpha^\dagger_{k+1}\left(C^{[4]}_{4;12|34}\right) + \alpha^\dagger_{k+1}\left(C^{[4]}_{4;12|45}\right)\\
&3\alpha_k\left(C^{[2]}_{3;1234}\right) + 3\alpha_k\left(C^{[2]}_{3;123x45}\right) + 3\alpha^\dagger_{k+1}\left(C^{[2]}_{3;123x45}\right) + 3\alpha_{k+5}\left(\beta\left(C^{[2]}_{3;123x45}\right)\right)\\
&+3\alpha^\dagger_k\left(\beta\left(C^{[2]}_{3;123x45}\right)\right) + \alpha_0\left(C^{[2]}_{3;12x34x56}\right) + \alpha^\dagger_1\left(C^{[2]}_{3;12x34x56}\right)\\
&+s_{k+1,k+2}\alpha_k\left(C^{[2]}_{2;12}\right) + s_{k+2,k+3}\alpha_{k+5}\left(\beta\left(C^{[2]}_{2;12}\right)\right) + s_{k+1,k+2,k+3}\alpha_k\left(C^{[2]}_{2;123}\right)\Bigg).
\end{aligned} \tag{5.63}$$

The procedure for computing analytic forms for these coefficients has been automated with the help of symbolic manipulations in FORM [71]. In particular the procedure does not require any prior algebraic manipulation of the tree level amplitude although the final form of the coefficient will depend on the form of the integrand. We have found it useful to choose a basis for the loop momenta that reflects the symmetry of the coefficient. Therefore adjacent massless legs were chosen where possible. However for the 2-mass easy box configuration it is much simpler to choose the two, non-adjacent, massless legs. The solution to the on-shell constraints in this case can be easily obtained from (4.5) using an exchange of $K_2 \leftrightarrow K_3$.

$$C^{[4]}_{4;123}(1^-,2^+,3^-,4^+,5^-,6^+) = \frac{2i}{\mathrm{tr}_5(\eta^a_{1,1+2|3},5,4,6)}$$
$$\times\left(\frac{\mathrm{tr}_5(\eta_{1,2},5,4,6)^2\mathrm{tr}_5(\eta_{3,2},5,4,6)^2[64]}{\langle 4|6|5]\langle 6|4|5][21][32]\mathrm{tr}_5(5,1,6,4)\mathrm{tr}_5(5,3,6,4)} - \frac{\langle 13\rangle^4\langle 5|46|5\rangle[64]}{s_{123}\langle 12\rangle\langle 23\rangle\langle 46\rangle}\right) \tag{5.64}$$

$$C^{[4]}_{4;12|34}(1^-,2^+,3^-,4^+,5^-,6^+) = \frac{2i\,\mathrm{tr}_5(\eta_{1,2},5,1+2,6)^2\mathrm{tr}_5(\eta_{3,4},5,1+2,6)^2}{s_{12}s_{34}s_{56}\langle 6|1+2|5]^2\mathrm{tr}_5(5,2,6,1)\mathrm{tr}_5(5,4,6,3)} \tag{5.65}$$

$$C^{[4]}_{4;12|45}(1^-,2^+,3^-,4^+,5^-,6^+) = C^{[4]}_{4;12|34}(1^-,2^+,5^-,4^+,3^-,6^+) \tag{5.66}$$

where we have defined complex momenta $\eta^\mu_{i,j} = \frac{1}{2}\langle i|\gamma^\mu|j]$ and $\eta^{a,\mu}_{i,j|k} = \frac{1}{2}\langle i|\gamma^\mu \not{j}|k\rangle$.

Following a similar automated procedure expressions for the triangle and bubble contributions have also been generated and have been checked numerically against the known results. However, since the expressions for these remaining coefficients are quite lengthy and not particularly illuminating, we include a `Mathematica` input file, `R6mpmpmpCoeffs.m`, containing the coefficients which follows the notation of the `S@M` package [63]. To present more compact forms ofthe triangle subtractions terms it was necessary to use a slightly expanded basis functions than was used in the five-point example:

$$D^a_{xyz}(\alpha,p,q) = -\alpha^2\langle xp\rangle[q^\flat|y|z\rangle + \alpha\left(\langle x|q^\flat y|z\rangle - \frac{q^2\langle x|py|z\rangle}{\langle p|q|p]}\right) + \langle xq^\flat\rangle[p|y|z\rangle, \tag{5.67}$$

$$D^b_{xyz}(\alpha,p,q) = -\alpha^2[xq^\flat]\langle p|y|z] + \alpha\left([x|q^\flat y|z] - \frac{q^2[x|py|z]}{\langle p|q|p]}\right) + [xp]\langle q^\flat|y|z], \tag{5.68}$$

$$D^{a'}_{xyz}(\alpha,\beta,p,q) = -\left(\frac{1}{2q\cdot p} + 2\alpha\beta\right)\langle xp\rangle[q^\flat|y|z\rangle + \beta\left(\langle x|q^\flat y|z\rangle - \frac{q^2\langle x|py|z\rangle}{2q\cdot p}\right), \tag{5.69}$$

$$D^{b'}_{xyz}(\alpha,\beta,p,q) = -\left(\frac{1}{2q\cdot p} + 2\alpha\beta\right)\langle xq^\flat\rangle\langle p|y|z] + \beta\left([x|q^\flat y|z] - \frac{q^2[x|py|z]}{2q\cdot p}\right), \tag{5.70}$$

with all other definitions given in equation (5.60). The bubble coefficients are then given by:

$$C^{[2]}_{2;12} = -\,iC^{[2],\mathrm{bub}}_{2;12} - \frac{1}{2}\sum_{\sigma=\pm} C^{[2],K_2=p_6}_{2;12} + C^{[2],K_2=p_5+p_6}_{2;12} + C^{[2],K_2=p_4+p_5+p_6}_{2;12} \tag{5.71}$$

$$C^{[2]}_{2;123}(K_1^\flat) = -\,iC^{[2],\mathrm{bub}}_{2;123}(K_1^\flat)$$
$$-\frac{1}{2}\sum_{\sigma=\pm} C^{[2],K_2=p_6}_{2;123}(K_1^\flat) + C^{[2],K_2=p_5+p_6}_{2;123}(K_1^\flat) + C^{[2],K_2=p_1+p_2}_{2;123}(-K_1^\flat) \tag{5.72}$$

where $\chi = p_1$ has been chosen for the basis in both cases.

Since the procedure for analytic extraction has been automated it has also been possible to generate unsimplified expressions for all other configurations with up to six gluons which have been checked numerically against existing results. These expressions are available from the author on request.

5.5 Remaining helicity configurations

Since the extraction is purely algebraic is also well suited a numerical extraction. One possible way to do this is to use the discrete Fourier projections to find the value of $C^{[4]}_4$.

From eq. (4.14) the box contribution can then be written as,

$$C_4^{[4]} = \frac{i}{2p_\mu} \sum_{\sigma=\pm} \sum_{k=0}^{p_\mu-1} \frac{1}{\mu_k^4} A_1 A_2 A_3 A_4(\bar{l}_1^\sigma, \mu_k), \tag{5.73}$$
$$\mu_k = \mu_\infty \exp\left(2\pi i \tfrac{k}{p_\mu}\right),$$

where μ_∞ is some large constant and $p_\mu + 1$ is the number of points around the circle of integration, on which the residue is evaluated. It is straightforward to write down equivalent expressions for the triangle,

$$\Rightarrow C_3^{[2]} = -\frac{1}{2p_\mu p_t} \sum_{\sigma} \sum_{k=0}^{p_\mu-1} \sum_{l=0}^{p_t-1} \frac{1}{\mu_k^2} A_1 A_2 A_3(\bar{l}_1^\sigma, t_l, \mu_k^2), \tag{5.74}$$
$$\mu_k = \mu_\infty \exp\left(2\pi i \tfrac{k}{p_\mu}\right), \qquad t_l = t_\infty \exp\left(2\pi i \tfrac{l}{p_t}\right), \tag{5.75}$$

and bubble terms,

$$C_2^{\text{bub}[2]} = -\frac{i}{p_\mu p_t p_y} \sum_{k=0}^{p_\mu-1} \sum_{l=0}^{p_t-1} \sum_{q=0}^{p_y-1} \sum_{s=0}^{2} \frac{Y_s}{\mu_k^2 y_q^s} A_1 A_2(\bar{l}_1(y_q, t_{l;-1}, \mu_k^2)) \tag{5.76}$$

$$C_2^{\text{tri}(K_3)[2]} = -\frac{1}{2p_\mu p_t} \sum_{\sigma=\pm} \sum_{k=0}^{p_\mu-1} \sum_{l=0}^{p_t-1} \sum_{s=1}^{3} \frac{T_s}{\mu_k^2 t_{l;1}^s} A_1 A_2 A_3(\bar{l}_1(y_\sigma, t_{l;1}, \mu_k^2)) \tag{5.77}$$
$$\mu_k = \mu_\infty \exp\left(2\pi i \tfrac{k}{p_\mu}\right) \qquad t_{l;\alpha} = t_\infty^\alpha \exp\left(2\pi i \tfrac{l}{p_t}\right) \qquad y_q = y_\infty \exp\left(2\pi i \tfrac{q}{p_y}\right).$$

Since the locations of the boundaries depend upon the order in which we take t and y large, we must take $\alpha = -1$ for the bubble terms and $\alpha = 1$ in the triangle subtraction terms.

The method of Fourier projections has already been shown to be extremely efficient in both the `BlackHat` code [31] and with the OPP technique [25]. In general one can arbitrarily increase the numerical accuracy by increasing the size of the circle, μ_∞, and the number of points $p_\mu \geq 4$. We note that, in order to find a stable numerical implementation at fixed precision, it would be necessary to make explicit subtractions of the poles from pentagon integrals [26, 53, 21]. This could be done in an analogous way to the treatment of the cut-constructible contributions in the `BlackHat` code by computing values of μ^2 which put the additional propagator on-shell [31].

We have tested our method for the direct extraction of the rational terms for the remaining one-loop gluon helicity amplitudes with up to six external legs using a numerical approach with discrete Fourier transforms. In order to compare with the known results it was sufficient to proceed without the subtraction of higher order residues in place of considering a large radius for the integration contour. The required tree level amplitudes have been generated using on-shell recursion relations literature [69, 70]. The rational terms have been checked against the known analytic results [37, 40, 38] and the numerical results of references [52 − 54]. We have also checked these results against expressions generated analytically through an automated implementation of the direct extraction method. In order to achieve numerical precision of at least 10^{-6} it was necessary to use a radius for each contour integral of 10^4 (i.e. the parameters $\mu_\infty, t_\infty, y_\infty$) and in some cases evaluate at a large number of points around the circle (i.e. $p_\mu, p_t, p_y \sim 10$). This was achieved using symbolic manipulation in FORM [71] and numerical procedures in Maple. In order to obtain a more stable implementation, subtraction of the residues of high point functions from the complex plane [31, 23] would allow setting all the radii to 1 and minimise the number of points required around the circle. It is expected that these improvements would considerably improve the speed of the algorithm but we leave such an implementation for future study.

6. Rational contributions to quark amplitudes

As an example of how the method can apply equally well to amplitudes with external fermions we re-compute the rational parts of the four-point process with a pair of massless quarks, $gg \to q\bar{q}$. The n-point colour ordered amplitude can be written as:

$$A_n^f = A_n^{[L]} + \frac{1}{N_c^2} A_n^{[R]} + \frac{N_f}{N_c} A_n^{L,[1/2]} + \frac{N_s}{N_c} A_n^{[0]} \tag{6.1}$$

$$= \left(1 + \frac{1}{N_c^2}\right) A_n^{[L]} - \frac{1}{N_c^2} A_n^{[SUSY]} - \left(\frac{N_f}{N_c} + \frac{1}{N_c^2}\right) (A_n^{[L,1/2]} + A_n^{[s]}) + \left(\frac{N_s - N_f}{N_c} - \frac{1}{N_c^2}\right) A_n^{[s]}. \tag{6.2}$$

The supersymmetric decomposition ensures that the only rational contributions come from the scalar term, $A_n^{[s]}$ (just as in the gluon case), and $A_n^{[L]}$ where the fermion line follows the shortest path through the loop (we refer the reader to [65] for definitions of the primitive amplitudes).

$$R_n^f = \left(1 + \frac{1}{N_c^2}\right) R_n^{[L]} + \left(\frac{N_s - N_f}{N_c} - \frac{1}{N_c^2}\right) R_n^{[s]} \tag{6.3}$$

Figure 5 shows the rational part of $A_4^{[L]}$ written in terms of the massive integral basis.

In the following calculation we show that it is possible to reconstruct the rational contributions to the left moving primitive amplitude by looking only at the transverse polarisation of the D-dimensional gluon. This can also be interpreted as introducing a massive scalar in the loop in an analogous way to the gluon examples. We then simply interpret the D-dimensional internal fermion as four-dimensional massive fermion.[5]

We now construct the tree-level structures using both massive scalars and massive fermions so that each cut has a uniform internal mass, μ^2.

$$V_3(1_q, 2_S, 3_Q) = i\bar{u}(p_1)v(p_3, \mu) \tag{6.4}$$

$$V_3(1_Q, 2_S, 3_q) = i\bar{u}(p_1, \mu)v(p_3) \tag{6.5}$$

[5] We do not present a formal proof of this fact since in general there is an ambiguity in the definition of the higher dimensional Clifford algebra. The definition is scheme dependent and working within the FDH scheme we simply note that there is no problem in treating the D-dimensional fermion as a massive one for the example presented here. Indeed D-dimensional massive fermions have been treated successfully within the FDH scheme in a recent paper [55]

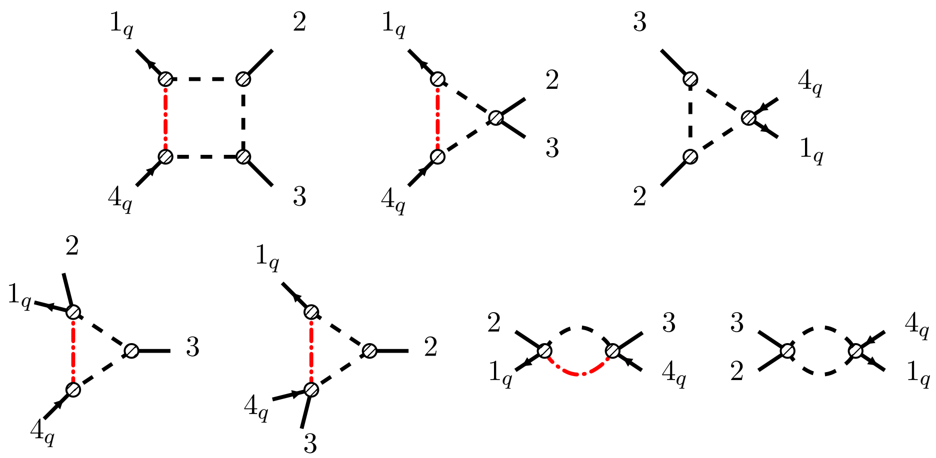


Figure 5: Integral basis for the rational part of the leading colour contribution to the $gg \to q\bar{q}$ amplitude, $R_4^{[L]}$.

where we can use the following representations for the fermion wave-functions:

$$\bar{u}_\pm(q,\eta) = \frac{\langle \eta \mp |(\not{q} + \mu)}{\langle \eta \mp | q^\flat \pm \rangle}, \qquad v_\pm(q,\eta) = \frac{(\not{q} - \mu)|\eta \mp\rangle}{\langle q^\flat \pm | \eta \mp \rangle} \tag{6.6}$$

The relevant four-point amplitudes are given by:

$$A_4(1_q, 2^+, 3_S, 4_Q) = \frac{i\bar{u}(p_1)v(p_4,\mu)\langle 1|3|2]}{\langle 2|3|2]\langle 12\rangle}, \tag{6.7}$$

$$A_4(1_q, 2^-, 3_S, 4_Q) = \frac{i\bar{u}(p_1)v(p_4,\mu)\langle 2|3|1]}{\langle 2|3|2][21]}, \tag{6.8}$$

$$A_4(1_Q, 2_S, 3^+, 4_q) = \frac{i\bar{u}(p_1,\mu)v(p_4)\langle 4|2|3]}{\langle 3|2|3]\langle 34\rangle}, \tag{6.9}$$

$$A_4(1_Q, 2_S, 3^-, 4_q) = \frac{i\bar{u}(p_1,\mu)v(p_4)\langle 3|2|4]}{\langle 3|2|3][43]}, \tag{6.10}$$

$$A_4(1_q^+, 2_S, 3_S, 4_q^-) = i\langle 4|2|1]\left(\frac{1}{s_{41}} + \frac{1}{2\langle 1|2|1]}\right). \tag{6.11}$$

The leading mass components of the seven integral coefficients, shown in figure 5, are easily computed with the techniques used for the gluon amplitudes in section 5. Firstly we note that for the leading colour primitive amplitude all box contributions are zero since the maximum power of μ that can appear in the product of the four three-point vertices is three. The first non-trivial coefficients are therefore the triangle coefficients which we present here for the three independent helicity configurations. The $+ + + -$ configuration is given by,

$$C_{3;12}^{[2]}(1_q^+, 2^+, 3^+, 4_{\bar{q}}^-) = \frac{i[12][23]}{2\langle 23\rangle[24]}, \tag{6.12}$$

$$C_{3;23}^{[2]}(1_q^+, 2^+, 3^+, 4_{\bar{q}}^-) = -\frac{i[23]^2}{2\langle 12\rangle[24]}, \tag{6.13}$$

$$C_{3;34}^{[2]}(1_q^+, 2^+, 3^+, 4_{\bar{q}}^-) = \frac{i\langle 14\rangle[12]}{2\langle 23\rangle\langle 13\rangle} + \frac{i\langle 14\rangle^2[12]}{\langle 12\rangle\langle 34\rangle\langle 13\rangle}, \tag{6.14}$$

$$C^{[2]}_{3;41}(1^+_q,2^+,3^+,4^-_{\bar q}) = -\frac{i\left(s_{14}\langle 13\rangle\langle 24\rangle + (s_{14}-4s_{24})\langle 12\rangle\langle 34\rangle\right)[32]}{2\langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 23\rangle[41]}, \tag{6.15}$$

$$\sum_{k=1}^{4} C^{[2]}_{3;k(k+1)}(1^+_q,2^+,3^+,4^-_{\bar q}) = -\frac{i[31](2s_{13}+s_{23})}{\langle 12\rangle\langle 23\rangle[41]}, \tag{6.16}$$

the $++--$ configuration by,

$$C^{[2]}_{3;12}(1^+_q,2^+,3^-,4^-_{\bar q}) = \frac{i\langle 34\rangle^2}{2\langle 12\rangle\langle 24\rangle}, \tag{6.17}$$

$$C^{[2]}_{3;23}(1^+_q,2^+,3^-,4^-_{\bar q}) = -\frac{i\langle 14\rangle\langle 23\rangle\langle 34\rangle}{2\langle 12\rangle^2\langle 24\rangle} + \frac{i\langle 34\rangle^2[21]}{\langle 23\rangle\langle 24\rangle[32]}, \tag{6.18}$$

$$C^{[2]}_{3;34}(1^+_q,2^+,3^-,4^-_{\bar q}) = C^{[2]}_{3;12}(1^+_q,2^+,3^-,4^-_{\bar q}), \tag{6.19}$$

$$C^{[2]}_{3;41}(1^+_q,2^+,3^-,4^-_{\bar q}) = -\frac{i\langle 14\rangle\langle 23\rangle[21]}{2\langle 12\rangle^2[31]}, \tag{6.20}$$

$$\sum_{k=1}^{4} C^{[2]}_{3;k(k+1)}(1^+_q,2^+,3^+,4^-_{\bar q}) = -\frac{is_{13}\langle 34\rangle[21]}{s_{23}\langle 24\rangle[43]}, \tag{6.21}$$

and finally the $+-+-$ configuration,

$$C^{[2]}_{3;12}(1^+_q,2^-,3^+,4^-_{\bar q}) = \frac{i\langle 24\rangle[41][43]}{2\langle 23\rangle[42]^2}, \tag{6.22}$$

$$C^{[2]}_{3;23}(1^+_q,2^-,3^+,4^-_{\bar q}) = -\frac{i[31]\left(s_{24}([31][42]+[21][43]) - s_{23}[21][43]\right)}{2\langle 23\rangle[21][32][42]^2}, \tag{6.23}$$

$$C^{[2]}_{3;34}(1^+_q,2^-,3^+,4^-_{\bar q}) = C^{[2]}_{3;12}(1^+_q,2^-,3^+,4^-_{\bar q}), \tag{6.24}$$

$$C^{[2]}_{3;41}(1^+_q,2^-,3^+,4^-_{\bar q}) = -\frac{i\langle 14\rangle\langle 23\rangle[31]}{2\langle 13\rangle^2[21]}, \tag{6.25}$$

$$\sum_{k=1}^{4} C^{[2]}_{3;k(k+1)}(1^+_q,2^+,3^+,4^-_{\bar q}) = \frac{i\langle 12\rangle[31]^3}{s_{23}[24]}. \tag{6.26}$$

There is only one non-zero bubble coefficient for all the four point fermion amplitudes considered here. This is the 23 channel for the $+++-$ helicity configuration for which we choose a basis defined by $K_1 = p_2 + p_3$ with $\chi = p_1$ ($\Rightarrow K_1^\flat = p_3$). This choice ensures that there is only one triangle subtraction term and the two integrands are:

$$C_{2;23}(1^+_q,2^+,3^+,4^-_{\bar q}) = \frac{2\mu^2[23]\langle 4|l_1|1]}{\langle 23\rangle\langle 2|l_1|2]}\left(\frac{1}{s_{41}} + \frac{1}{2\langle 1|l_1|1]}\right) \tag{6.27}$$

$$C^{\mathrm{tri};K_3=p_1}_{2;23}(1^+_q,2^+,3^+,4^-_{\bar q}) = \frac{2\mu^2[23]\langle 4|l_1|1]}{\langle 23\rangle\langle 2|l_1|2]}. \tag{6.28}$$

Since $T_0 = 0$, the triangle subtraction term will vanish after substitution and expansion of the loop momentum. The second term in the pure bubble is also suppressed by an additional power of the loop momentum and therefore vanishes after taking the large y limit. The single remaining term is then gives the final value for the coefficient:

$$C^{[2]}_{2;23}(1^+_q,2^+,3^+,4^-_{\bar q}) = -\frac{2i\langle 4|2-3|1]}{\langle 23\rangle^2 s_{23}} \tag{6.29}$$

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The rational contributions to the left moving primitive amplitudes for the process $q\bar{q}gg$ are therefore given by:

$$R_4^{[L]}(1_q,2,3,4_{\bar{q}}) = -\frac{1}{2}\sum_{k=1}^{4} C_{3;k(k+1)}^{[2]}(1_q,2,3,4_{\bar{q}}) - \frac{s_{23}}{6}C_{2;23}^{[2]}(1_q,2,3,4_{\bar{q}}) \tag{6.30}$$

Summing all of these components together can be quickly shown to match the results for four point amplitudes [65, 74]:

$$R_4^{[L]}(1_q^+,2^+,3^+,4_{\bar{q}}^-) = -\frac{i\langle 41\rangle[13]}{2\langle 12\rangle\langle 23\rangle}\left(1+\frac{2s_{12}}{3s_{23}}\right), \tag{6.31}$$

$$R_4^{[L]}(1_q^+,2^+,3^-,4_{\bar{q}}^-) = \frac{1}{2}A_4^{(0)}(1_q^+,2^+,3^-,4_{\bar{q}}^-), \tag{6.32}$$

$$R_4^{[L]}(1_q^+,2^-,3^+,4_{\bar{q}}^-) = -\frac{1}{2}\left(1+\frac{s_{23}}{s_{13}}\right)A_4^{(0)}(1_q^+,2^-,3^+,4_{\bar{q}}^-). \tag{6.33}$$

The tree amplitudes above are the well known MHV amplitudes:

$$A_4^{(0)}(1_q^+,2^+,3^-,4_{\bar{q}}^-) = \frac{i\langle 34\rangle^3\langle 31\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}, \tag{6.34}$$

$$A_4^{(0)}(1_q^+,2^-,3^+,4_{\bar{q}}^-) = \frac{i\langle 24\rangle^3\langle 21\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \tag{6.35}$$

7. Conclusions

We have presented a general method for extracting rational contributions to gauge-theory scattering amplitudes using D-dimensional unitarity techniques [26, 53]. Exchanging the D-dimensional cuts for four dimensional massive cuts, we then used simple complex analysis to express the D-dimensional integral coefficients as contour integrals over a complex mass, μ, evaluated on a circle at infinity. Here we have gone beyond previous approaches [21, 26, 53] by making maximum use of the complex behaviour of the corresponding contour integrals. We have shown that it is not necessary to compute the pentagon contributions explicitly, thereby reducing the computation of rational terms to the evaluation of generalised massive cuts of box, triangle and bubble functions in the large mass-limit. The formalism leads directly to compact analytic expressions for the rational terms.

To demonstrate the method we have computed analytic expressions for the rational parts of one-loop gluon amplitudes using a massive scalar loop. This has been tested by re-computing gluon helicity amplitudes with up to six external legs. We have also shown that the method can also be implemented numerically using a discrete Fourier transform. The implementation was sufficient to evaluate all helicity configurations with up to six external gluons but we leave a detailed analysis of accuracy and speed to future work.

We have also presented a simple example of how the method can be used to compute amplitudes with massless external fermions, considering the leading colour primitive amplitude for the process $qq \to gg$. Here we show that the rational terms can be extracted using tree amplitudes where a massive fermion couples to a massive scalar inside the loop.

Since the procedure is based on looking at four-dimensional cuts it is reasonable to expect that one can obtain a similar level of computational speed in comparison to calculations of the cut-constructible terms [47, 31], though further analysis of the subtraction terms is expected to be necessary. It does have the benefit over on-shell recursive techniques that it avoids problems of unknown factorisation properties in the complex plane and therefore applies generally to any process, in particular to those with internal masses.

Although the present paper concentrates on applications to amplitudes with massless external particles the main part of the procedure applies equally well to the case of massive ones. In this case the bases for the loop momentum in each of the cuts would have to be modified slightly as described in reference [30].

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A. μ^2 dependence of the integral coefficients

In this appendix we use a simple power counting argument to demonstrate the maximum power of μ^2 that can appear in the n-point gluon amplitudes fits the format described in section 3. Although the final result is assured for general amplitudes in any gauge theory the following analysis sheds light on the analytic expansions of the Inf expansions, showing that only a few terms in the expansion contribute to the final rational terms. This information can then be used for efficient computations of the boundary behaviour. For the form to hold we must show that the leading dependence on μ appearing in any massive box cut is μ^4 and μ^2 in any triangle or bubble. This property is shown to hold in both the OPP/GKM formalism [26, 53] using a simple argument from the dimension of possible tensor structures. It is this argument that allows the parametrisation of the cut in the GKM approach and so we see that it is related to the decompositions of the Inf_μ components of the cut given in equations (4.13),(4.18) and the analogous relation for the bubble coefficients. In the analytic approach of Britto et al. the μ-dependence can be determined through explicit analysis of the spinor integrals [21]. Here we use a slightly different approach to justify these statements, appealing to properties of the tree level amplitudes.

We begin by considering a general quadruple cut given by the product of four tree amplitudes. Schematically the loop momentum can be written as,

$$l(\mu^2) = \alpha_1 + \alpha_2 c + \frac{\alpha_3}{c} - \frac{\mu^2 \alpha_4}{c} \tag{A.1}$$

the solution for c is given by equation (4.5) from which it is clear that c scales as $\sqrt{\mu^2}$. It is then clear that the loop momentum given above scales as $\sqrt{\mu^2}$ in the large μ limit.

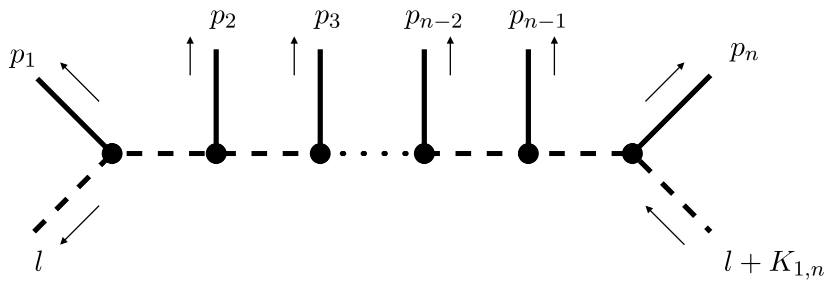

Figure 6: The momentum flow of a massive scalar through a generic tree level Feynman graph.

The next step is to consider the dependence of a generic tree amplitude on the loop momentum. For clarity we restrict ourselves to considering a single massive scalar loop although it should also be possible to apply a similar argument to more general cases. The tree level amplitude can be characterised by considering a massive scalar flowing through a general Feynman graph as shown in figure 6. The gluon-scalar-scalar vertex is simply proportional to $l \cdot \varepsilon_k$ and each of the propagators will scale as,

$$\frac{1}{(l+K)^2-\mu^2} = \frac{1}{2(l\cdot K)+K^2}.$$

The maximum dependence on l will come from graphs with the maximum number of vertices. Such graphs will have n vertices but will also have $n-1$ propagators, from which we can deduce:

$$A_s^{(0)}(l) \overset{l\to\lambda l}{\underset{\lambda\to\infty}{\to}} \lambda A_s^{(0)}(l). \tag{A.2}$$

Taken together with the scaling of the loop momentum, this is sufficient to show that the product of four tree amplitudes scales at most as μ^4 in the limit $\mu\to\infty$.

For the triangle coefficients we have to consider the expansion in t in order to observe the expected behaviour. Here the loop momentum is given as

$$l(\mu^2,t) = \alpha_1 + \alpha_2 t + \frac{\alpha_3}{t} + \frac{\mu^2\alpha_4}{t}. \tag{A.3}$$

We have that in any triple cut the maximum power of t that can appear will be three, which follows from the argument given above for the scaling of the tree level amplitudes. More specifically we have maximum difference of three powers of the loop momenta between the numerator and denominator and so we can write the boundary behaviour of a general triple cut as:

$$\mathrm{Inf}_{\mu^2}[\mathrm{Inf}_t[C_3]] = \frac{P_n(a)}{P_{n-3}(b)} \tag{A.4}$$

$$P_n(x) = \sum_{m=0}^{n}\sum_{k=-n}^{n-2m} t^k \mu^{2m} x_{n-k,m} \tag{A.5}$$

In order to encode all the mass dependence in the coefficients a,b we have first re-written all explicit mass dependence in the tree amplitudes as $\mu^2 = l_1\cdot l_1$. After expansion, the t^0

component of C_3 is,

$$\begin{aligned}\mathrm{Inf}_t[C_3]|_{t^0} =&\mu^2\left(-\frac{a_{1,0}b_{2,1}}{b_{0,0}^2}-\frac{a_{0,0}b_{3,1}}{b_{0,0}^2}+2\frac{a_{0,0}b_{1,0}b_{2,1}}{b_{0,0}^3}-\frac{b_{1,0}a_{2,1}}{b_{0,0}^2}+\frac{a_{3,1}}{b_{0,0}}\right)\\ &-\frac{a_{1,0}b_{2,0}}{b_{0,0}^2}+2\frac{a_{0,0}b_{1,0}b_{2,0}}{b_{0,0}^3}-\frac{a_{0,0}b_{3,0}}{b_{0,0}^2}+\frac{a_{3,0}}{b_{0,0}}-\frac{b_{1,0}a_{2,0}}{b_{0,0}^2}+\frac{b_{1,0}^2a_{1,0}}{b_{0,0}^3}-\frac{b_{1,0}^3a_{0,0}}{b_{0,0}^4}\end{aligned} \tag{A.6}$$

which shows that the triangle coefficients have the expected behaviour.

Finally we use a similar argument to prove that the bubble coefficients also scale at most as μ^2. The loop momentum in this case is

$$l(\mu^2,t,y)=\alpha_1\frac{y^2}{t}+\alpha_2\frac{y}{t}+\alpha_3 y+\alpha_4 t+\alpha_5\frac{\mu^2}{t}+\alpha_6 \tag{A.7}$$

We then write down the boundary behaviour of the most general double cut integrand as a ratio of two polynomials,

$$\mathrm{Inf}_\mu[\mathrm{Inf}_t[\mathrm{Inf}_y[[C_2]]]=\frac{P'_n(a)}{P'_{n-2}(b)} \tag{A.8}$$

$$P'_n(x)=\sum_{m=0}^{n}\sum_{k=-n}^{n-2m}\sum_{l=\max(0,-k-m)}^{n-k-2m}\mu^{2m}t^k y^l x_{2n-l,k+n,m} \tag{A.9}$$

Again, using the argument on the scaling of the tree-level amplitudes, we see that the polynomial in the numerator has two powers more than the one in the numerator. The series expansion of C_2 then gives us the following expression for the pure bubble cut:

$$\begin{aligned}\mathrm{Inf}_t[\mathrm{Inf}_y[C_2]]|_{t^0,y\to Y_i} =& Y_2\left(\frac{a_{2,2,0}}{b_{0,0,0}}-\frac{a_{0,0,0}\,b_{2,2,0}}{{b_{0,0,0}}^2}-\frac{a_{1,1,0}\,b_{1,1,0}}{{b_{0,0,0}}^2}+\frac{a_{0,0,0}\,{b_{1,1,0}}^2}{{b_{0,0,0}}^3}\right)\\
&+Y_1\left(\frac{a_{2,2,0}}{b_{0,0,0}}-\frac{a_{0,0,0}\,b_{2,2,0}}{{b_{0,0,0}}^2}-\frac{a_{1,1,0}\,b_{1,1,0}}{{b_{0,0,0}}^2}+\frac{a_{0,0,0}\,{b_{1,1,0}}^2}{{b_{0,0,0}}^3}\right)\\
&+\mu^2\left(2\,\frac{a_{0,0,0}\,b_{2,0,1}\,b_{2,2,0}}{{b_{0,0,0}}^3}+2\,\frac{a_{0,0,0}\,b_{1,1,0}\,b_{3,1,1}}{{b_{0,0,0}}^3}-\frac{a_{2,0,1}\,b_{2,2,0}}{{b_{0,0,0}}^2}+\frac{a_{4,2,1}}{b_{0,0,0}}-\frac{a_{0,0,0}\,b_{4,2,1}}{{b_{0,0,0}}^2}-\frac{a_{1,1,0}\,b_{3,1,1}}{{b_{0,0,0}}^2}\right.\\
&\left.\quad-\frac{a_{3,1,1}\,b_{1,1,0}}{{b_{0,0,0}}^2}-\frac{a_{2,2,0}\,b_{2,0,1}}{{b_{0,0,0}}^2}+\frac{a_{2,0,1}\,{b_{1,1,0}}^2}{{b_{0,0,0}}^3}-3\,\frac{a_{0,0,0}\,{b_{1,1,0}}^2b_{2,0,1}}{{b_{0,0,0}}^4}+2\,\frac{a_{1,1,0}\,b_{1,1,0}\,b_{2,0,1}}{{b_{0,0,0}}^3}\right)\\
&-\frac{a_{0,0,0}\,b_{4,2,0}}{{b_{0,0,0}}^2}-\frac{a_{1,1,0}\,b_{3,1,0}}{{b_{0,0,0}}^2}-\frac{a_{1,0,0}\,b_{3,2,0}}{{b_{0,0,0}}^2}-\frac{a_{2,0,0}\,b_{2,2,0}}{{b_{0,0,0}}^2}-\frac{a_{2,1,0}\,b_{2,1,0}}{{b_{0,0,0}}^2}-\frac{a_{2,2,0}\,b_{2,0,0}}{{b_{0,0,0}}^2}\\
&-\frac{a_{3,1,0}\,b_{1,1,0}}{{b_{0,0,0}}^2}+\frac{a_{2,0,0}\,{b_{1,1,0}}^2}{{b_{0,0,0}}^3}+\frac{a_{2,2,0}\,{b_{1,0,0}}^2}{{b_{0,0,0}}^3}-\frac{a_{3,2,0}\,b_{1,0,0}}{{b_{0,0,0}}^2}+\frac{a_{4,2,0}}{b_{0,0,0}}+6\,\frac{a_{0,0,0}\,{b_{1,0,0}}^2{b_{1,1,0}}^2}{{b_{0,0,0}}^5}\\
&+2\,\frac{a_{0,0,0}\,b_{1,0,0}\,b_{3,2,0}}{{b_{0,0,0}}^3}+2\,\frac{a_{0,0,0}\,b_{1,1,0}\,b_{3,1,0}}{{b_{0,0,0}}^3}+2\,\frac{a_{1,1,0}\,b_{1,0,0}\,b_{2,1,0}}{{b_{0,0,0}}^3}+2\,\frac{a_{1,0,0}\,b_{1,0,0}\,b_{2,2,0}}{{b_{0,0,0}}^3}\\
&+2\,\frac{a_{1,0,0}\,b_{1,1,0}\,b_{2,1,0}}{{b_{0,0,0}}^3}+2\,\frac{a_{0,0,0}\,b_{2,2,0}\,b_{2,0,0}}{{b_{0,0,0}}^3}+2\,\frac{a_{1,1,0}\,b_{1,1,0}\,b_{2,0,0}}{{b_{0,0,0}}^3}-3\,\frac{a_{0,0,0}\,{b_{1,0,0}}^2b_{2,2,0}}{{b_{0,0,0}}^4}\\
&-3\,\frac{a_{0,0,0}\,{b_{1,1,0}}^2b_{2,0,0}}{{b_{0,0,0}}^4}-6\,\frac{a_{0,0,0}\,b_{1,0,0}\,b_{1,1,0}\,b_{2,1,0}}{{b_{0,0,0}}^4}-3\,\frac{a_{1,1,0}\,{b_{1,0,0}}^2b_{1,1,0}}{{b_{0,0,0}}^4}+2\,\frac{a_{2,1,0}\,b_{1,1,0}\,b_{1,0,0}}{{b_{0,0,0}}^3}\\
&+\frac{a_{0,0,0}\,{b_{2,1,0}}^2}{{b_{0,0,0}}^3}-3\,\frac{a_{1,0,0}\,{b_{1,1,0}}^2b_{1,0,0}}{{b_{0,0,0}}^4}\end{aligned} \tag{A.10}$$

Once we substitute the functions Y_i from eq. (4.26) we see that the coefficient scales as μ^2. For the triangle subtraction terms the parametrised loop momentum is the same as in the scalar triangle coefficient but this time we are interested in the coefficients of t^3, t^2 and t:

$$\begin{aligned}\mathrm{Inf}_t[C_2^{[\mathrm{tri}]}]|_{t^i=T_i} =&T_3\left(\frac{a_{0,0}}{b_{0,0}}\right)+T_2\left(\frac{a_{1,0}}{b_{0,0}}-\frac{a_{0,0}b_{1,0}}{b_{0,0}^2}\right)\\ &+T_1\left(\mu^2\left(-\frac{a_{0,0}b_{2,1}}{b_{0,0}^2}+\frac{a_{2,1}}{b_{0,0}}\right)+\frac{a_{2,0}}{b_{0,0}}-\frac{a_{0,0}b_{2,0}}{b_{0,0}^2}-\frac{b_{1,0}a_{1,0}}{b_{0,0}^2}+\frac{b_{1,0}^2a_{0,0}}{b_{0,0}^3}\right)\end{aligned}\tag{A.11}$$

which after we use equations (4.31)–(4.33) shows that the full bubble coefficient scales as expected.

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